ANSWERS

Answer 1

(a) S_1, S_2, S_4, S_5 with value $21+22+31+32 = 106$.

(b)

$$\begin{aligned} \min \quad & 21x_1 + 22x_2 + 23x_3 + 31x_4 + 32x_5 + 33x_6 \\ s.t. \quad & x_1 + x_2 \geq 1 \\ & x_1 + x_3 \geq 1 \\ & x_2 + x_3 \geq 1 \\ & x_4 + x_5 \geq 1 \\ & x_4 + x_6 \geq 1 \\ & x_5 + x_6 \geq 1 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \in \{0,1\}. \end{aligned}$$

(c) $x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 1/2$.

(d)

$$\begin{aligned} \max \quad & y_1 + y_2 + y_3 + y_4 + y_5 + y_6 \\ s.t. \quad & y_1 + y_2 \leq 21 \\ & y_1 + y_3 \leq 22 \\ & y_2 + y_3 \leq 23 \\ & y_4 + y_5 \leq 31 \\ & y_4 + y_6 \leq 32 \\ & y_5 + y_6 \leq 33 \\ & y_1, y_2, y_3, y_4, y_5, y_6 \geq 0. \end{aligned}$$

(e) All dual constraints are tight so all sets are chosen. The value is 162.

Answer 2

The optimal solution must have $k = 2$ since the optimal value is at most $1000/k$ (which is less than 400 for k larger than 2). That means, the graph can be partitioned into independent sets S_1, S_2, one with 400 and the other with 600 vertices. We can find these sets using the greedy 2-coloring algorithm: Take any vertex and assign it to S_1. Then assign all its neighbours to S_2. Next assign all the neighbours of these vertices to S_1, and so on. Since the graph is connected all vertices will be assigned to one of the two sets and this partition is unique, that means, it must be the optimal partition.

Answer 3

(a) $1/2+3/4+3/4+3/4 = 2.75$.

(b) Let Z be the number of clauses satisfied.
$E[Z \mid x_1 = \text{True}] = 1+1+1/2+3/4 = 3.25$.
$E[Z \mid x_1 = \text{False}] = 0+1/2+1+3/4 = 2.25$.
So take $x_1 =$True.

$E[Z \mid x_1 = \text{True}, x_2 = \text{True}] = 1+1+1/2+1 = 3.5.$
$E[Z \mid x_1 = \text{True}, x_2 = \text{False}] = 1+1+1/2+1/2 = 3.$
So take x_2 =True.
$E[Z \mid x_1 = \text{True}, x_2 = \text{True}, x_3 = \text{True}] = 1+1+0+1 = 3.$
$E[Z \mid x_1 = \text{True}, x_2 = \text{True}, x_3 = \text{False}] = 1+1+1+1 = 4.$
So take x_3 =False.

Answer 4

(a) 11 (Vertex 5 is left out. The penalty for this is 2. The cost of the tree is $3+3+3 = 9$.)

(b) No. For $i = 4$ and $S = \{3,4\}$) the cut constraint in the LP does not hold: $1/3+1/3 < 1$.

(c) For each terminal, take a large penalty M and for each other vertex take penalty zero. For example, take M bigger than the sum of all edge costs.
Explanation: An optimal prize collecting Steiner tree must include all terminals since leaving out any of these cannot be optimal (since the cost would be more than cost of a tree on all vertices). There is no penalty for including any non-terminal in the tree. Hence, this is exactly the standard Steiner tree problem.

Answer 5

$A = 1$, $B = 0$, $C = M$, where M is a large number. (M larger than the number of sets will do.)
Explanation: In an optimal solution for the UFL, a client j can only connect to a facility i if $j \in S_i$ since otherwise the cost is too large (even more than opening all facilities). The cost of the optimal UFL is exactly the number of sets in an optimal set cover.
(N.B. It does not work to take $B = 1/|S_i|$.)

Answer 6

Let k be the job that completes last. Let C_k be its completion time and S_k its start time. Then

$$C_k = S_k + p_k \leq \frac{1}{m}\sum_{j \neq k} p_j + p_k = \frac{1}{m}\sum_{j} p_j + \frac{m-1}{m} p_k.$$

Next, use the given inequality for p_k and the that $\text{OPT} \geq \sum_j p_j/m$.

$$.... \leq \frac{1}{m}\sum_{j} p_j + \frac{1}{3m}\sum_{j} p_j \leq \text{OPT} + \text{OPT}/3.$$

Answer 7

(a) 400

(b) a, c, b.

(c) For a: $1 \cdot 100 + 1 \cdot 100 + 1 \cdot 100 + 1 \cdot 100 + 0 \cdot 1 = 400$
For b: $\frac{2}{5} \cdot 100 + \frac{2}{5} \cdot 100 + \frac{2}{5} \cdot 100 + \frac{2}{5} \cdot 100 + \frac{2}{5} \cdot 1 = 160.4$
For c: $\frac{4}{5} \cdot 100 + \frac{4}{5} \cdot 100 + \frac{4}{5} \cdot 100 + \frac{4}{5} \cdot 100 + \frac{4}{5} \cdot 1 = 320.8$