Answers

1. **(a)** The optimal value is 7.

(b)

$$
\begin{array}{ll}
\min & 4x_1 + 3x_2 + x_3 + 2x_4 + 2x_5 \\
s.t. & x_1 + x_2 \geq 1 \\
 & x_1 + x_3 \geq 1 \\
 & x_1 + x_4 \geq 1 \\
 & x_1 + x_5 \geq 1 \\
 & x_2 + x_4 \geq 1 \\
 & x_2 + x_5 \geq 1 \\
 & x_1, x_2, x_3, x_4, x_5 \in \{0,1\} \qquad .
\end{array}
$$

A solution of value 6: All x-variables equal to $1/2$.

(c)

$$
\begin{array}{ll}
\max & y_{12} + y_{13} + y_{14} + y_{15} + y_{24} + y_{25} \\
s.t. & y_{12} + y_{13} + y_{14} + y_{15} + \leq 4 \\
 & y_{12} + y_{24} + y_{25} \leq 3 \\
 & y_{13} \leq 1 \\
 & y_{14} + y_{24} \leq 2 \\
 & y_{15} + y_{25} \leq 2 \\
 & y_{12}, y_{13}, y_{14}, y_{15}, y_{24}, y_{25} \geq 0
\end{array}
$$

A solution of value 6: All y-variables equal to 1.

(d)

Primal: All x-variables are $1/2$. The value is $1/2$ times the sum of the degrees, which is $|E|$. (The sum of the degrees is exactly twice the number of edges.)

Dual: All y-variables equal to 1. The value is $|E|$.

Since the value of the LP-solution is the same as the value of the dual-solution, both solutions are optimal.

2. An ILP:

$$
\begin{array}{lll}
\min & \sum_{i \in V} x_i & \\
s.t. & \sum_{i \in t} x_i \geq 2 & \text{for all triangles } t \in T. \\
 & x_i \in \{0,1\} & \text{for all } i \in V.
\end{array}
$$

In the LP-relaxation we take $0 \leq x_i \leq 1$.

Algorithm: Solve the LP. Add vertex i to solution S if $x_i^* \geq 0.5$.

The algorithm runs in polynomial time since the LP can be solved in polynomial time.

The solution is feasible since for each constraint (triangle) there are always at least two x-variables with value at least 0.5.

The approximation ratio is at most 2 since the x-variables are increased by at most a factor 2. More precisely: Denote the rounded solution by $\hat{x}$. That means: $\hat{x}_i = 1$ if $x_i^* \geqslant 1/2$ and $\hat{x}_i = 0$

otherwise. In either case, $\hat{x}_i \leqslant 2x_i^*$. The value of the solution found is

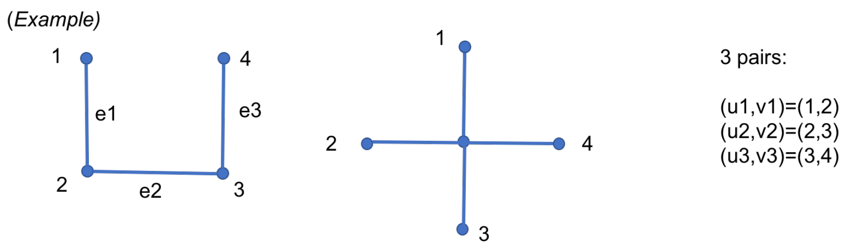
$$|S| = \sum_{i \in V} \hat{x}_i \leqslant 2 \sum_{i \in V} x_i^* = 2Z_{LP}^* \leqslant 2Z_{ILP}^* = 2\text{OPT}.$$

3. Let S_k be the start time of job k and let p_k be its length. Then the length of the schedule is $S_k + p_k$. We know that $S_k \leq \text{OPT}$ since there is no idle time before time S_k. (Otherwise, the LPT algorithm would have started job k earlier.) Also $p_k \leq S_k/4$ since there are at least 5 jobs on each machine and job k is the smallest on its machine.

Hence, $S_k + p_k \leq (5/4)S_k \leq (5/4)\text{OPT}$.

4. **(a)** Given an instance of VC define a pair (u_i, v_i) for each edge (u_i, v_i). The optimal value of 1of2 TSP is exactly the optimal value of the VC instance.
(So the optimal value of VC can be computed by computing the optimal value of the 1of2 TSP instance.)

(b)



Given an instance of VC define a star on $n+1$ vertices: One vertex has degree n and the other vertices have degree 1. For each vertex of the graph there is one leaf. For each edge (u, v) of the graph define a corresponding pair in the tree. See the example. The optimal value of 1of2 TSP is exactly twice the optimal value of the VC instance.
(So the optimal value of VC can be computed by computing the optimal value of the 1of2 TSP instance.)

5. **(a) Algorithm**
Step 1: Construct a preemptive schedule using the SRPT rule.
Step 2: Schedule the jobs non-preemptively and as early as possible in the order of the completion times in the SRPT schedule.

(b)

$$\begin{aligned} \text{(LP) min} \quad & Z = \sum_{j=1}^{n} w_j C_j & \\ s.t. \quad & C_j \geqslant r_j + p_j & \text{for all jobs } j \\ & \sum_{j \in S} p_j C_j \geqslant \tfrac{1}{2}(\textstyle\sum_{j \in S} p_j)^2 & \text{for all sets } S \subseteq \{1, \ldots, n\} \end{aligned}$$

Algorithm
Step 1: Solve the LP.

Step 2: Schedule the jobs non-preemptively and as early as possible in the order of the LP-completion times.

6. (a)

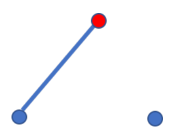
$$\begin{aligned}
\min \quad & \sum_{e\in E} c_e x_e + \sum_{i\in V} \pi_i(1-y_i) \\
s.t. \quad & \sum_{e\in\delta(S)} x_e \geqslant y_i, && \text{for all } i,S \text{ with } i\in S\subseteq V-r, \\
& y_r = 1, \\
& x_e \in \{0,1\}, && \text{for all } e\in E, \\
& y_i \in \{0,1\}, && \text{for all } i\in V.
\end{aligned}$$

(b)

$$\begin{aligned}
\min \quad & 4x_{12}+7x_{13}+7x_{23}+5(1-y_2)+6(1-y_3) \\
s.t. \quad & x_{12}+x_{23} \geq y_2 \\
& x_{12}+x_{13} \geq y_2 \\
& x_{12}+x_{13} \geq y_3 \\
& x_{13}+x_{23} \geq y_3 \\
& y_1 = 1, \\
& x_{12},x_{13},x_{23} \in \{0,1\} \\
& y_1,y_2,y_3 \in \{0,1\}
\end{aligned}$$

.

The optimal solution (The value is $4+6=10$):



(c) Given a candidate solution of an LP, it tells wether or not it satisfies all constraints. If not, then it returns a constraint that does not hold.

(d) For each vertex i, compute the value of a minimum capacity cut between r and i, where the x-values are the capacities. If for some i the minimum cut has value less than y_i then a violated constraint is found. If the value is at least y_i for all i then the solution is feasible.

A minimum cut can be found in polynomial time and we need to compute at most $n-1$ of them.

7. **(a)** For $i = 1,2,3$:

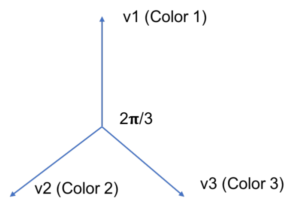
Cut i: All vertices with color i on one side and the other two colors on the other side.

Each edge is in exactly two of those cuts. So in total, the 3 cuts contain exactly twice the number of edges. At least one of the cuts contains at least $2/3$ of the edges.

(b)

$$\begin{array}{llll} \text{(VP)} & \min & \lambda & \\ & s.t. & v_i \cdot v_j \leqslant \lambda, & \text{for all } (i,j) \in E \\ & & v_i \cdot v_i = 1, v_i \in \mathbb{R}^n & \text{for all } i \in V. \end{array}$$

Claim: The optimal value of the VP is at most -0.5.
Proof: See figure. If we assign to each vertex of color k (for $k = 1,2,3$) the vector as in the figure, then the value of the solution is $\cos(2\pi/3) = -0.5$. So the optimal value is at most -0.5.



Algorithm: Solve the VP. Then take a unit vector r at random. One side of the cut are all vertices i for which $v_i \cdot r \geq 0$. The other side are all vertices i for which $v_i \cdot r < 0$.

The probability that an edge (i,j) is in the cut is exactly φ_{ij}/π, where φ_{ij} is the angle between vectors v_i and v_j in the optimal VP-solution. From $v_i \cdot v_j \leq -0.5$ it follows that $\varphi_{ij} \geq 2\pi/3$. Hence, the probability that an edge (i,j) is in the cut is at least $2/3$. In total, the expected number of edges in the cut is at least $(2/3)|E|$.