Exam Combinatorial Optimization, 25 October 2017, 8.45-11.30

- It is not allowed to use any books, notes, or calculator. Just pen and paper.
- Solutions will be on Canvas after the exam.
- The table shows the maximum number of points per (sub-)question.

1a	1b	1c	2a	2b	2c	2d	3	4a	4b	4c	4d	5a	5b	5c	6a	6b	6c	6d	Σ
2	2	2	2	3	3	2	5	3	1	2	5	2	2	3	3	2	1	5	50

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1. Consider the following instance of the unweighted set cover problem. The elements are $E = \{1,2,3,4,5\}$ and the subsets are $S_1 = \{2,3,4,5\}$, $S_2 = \{1,3,4,5\}$, $S_3 = \{1,2,4,5\}$, $S_4 = \{1,2,3,5\}$, and $S_5 = \{1,2,3,4\}$.

(a) Write down the ILP for this set cover instance. What is the optimal value?

(b) Give a solution to the LP-relaxation with value strictly less than the optimal ILP value.

(c) Write down the dual of the LP for this instance.

2. **(a)** Formulate the ILP for the unweighted vertex cover problem.

(b) Show that for any integer $n \geq 2$ there is a graph on n vertices such that

$$\text{OPT}_{VC}/\text{OPT}_{LP} \geq 2 - 2/n,$$

where OPT_{VC} denotes the optimal value of the vertex cover instance and OPT_{LP} denotes the optimal value of the LP-relaxation.

(c) Formulate the ILP for the independent set problem.

(d) Show that for any any integer $n \geq 2$ there is a graph on n vertices such that

$$\text{OPT}_{LP}/\text{OPT}_{IS} \geq n/2,$$

where OPT_{IS} denotes the optimal value of the independent set instance and OPT_{LP} denotes the optimal value of the LP-relaxation.

3. There are several constant-factor approximation algorithms known for the metric traveling salesman problem. However, for non-metric instances (that means, the triangle inequality does not need to hold) there cannot exist any constant-factor approximation algorithm, assuming $P \neq NP$. Prove this by giving a reduction form some well-known NP-complete problem.

4. **(a)** Section 4.2 gives an approximation algorithm for the scheduling problem $1|r_j|\sum w_jC_j$. Give the LP formulation that was used in that algorithm.

(b) Now consider the following instance with 5 jobs. All jobs have unit processing time and unit weight: $p_j = w_j = 1$ for $j = 1\dots 5$. All release times are the same: $r_j = 1$ for $j = 1\dots 5$. What is the optimal value? (Not for the LP but the real optimum.)

(c) If we would write down the LP for the instance given in (b) then what is the number of constraints of the LP? Explain your counting.

(d) Is the following solution feasible for the LP? If not, then give a violated constraint. If it is feasible then argue why it is feasible and do this without explicitly verifying *all* of the LP-constraints.

$$C_1 = 3.2,\ C_2 = 3.1,\ C_3 = 3,\ C_4 = 2.9,\ C_5 = 2.8.$$

5. **(a)** Chapter 5 gives several approximation algorithms for the maximum satisfiability problem. There is a very simple randomized $1/2$-approximation algorithm for this problem. Give the algorithm and prove that the approximation factor is indeed at least $1/2$.

(b) If you would apply the $1/2$-approximation algorithm to the following instance then what is the expected number of clauses satisfied?

$$(x_1 \vee x_2),\ (x_1 \vee \overline{x_2}),\ (\overline{x_1} \vee x_2),\ (\overline{x_1} \vee \overline{x_2}),\ (x_1 \vee x_2 \vee x_3)$$

(c) Some of the approximation algorithms (in chapter 5) for the maximum satisfiability problem use LP-rounding. Give that LP formulation.
(Give the general formulation, not just for the example, and use the following notation. Let w_j be the weight of clause j. Let P_j be the positive variables of clause j and let N_j be the negated variables of clause j.)

6. **(a)** Give the Quadratic Program for the maximum cut problem. (Either give the general formulation or give it for the example below.)

(b) Give the Vector Program relaxation of the formulation given in (a).

(c) What is the value of the maximum cut in the example? Argue why that is the maximum value.

(d) Show that the optimal solution to the vector program for this instance has a value that is strictly larger than the value of the maximum cut. *Hint: the graph is 3-colorable.*