Answers

1. Consider the following instance of the unweighted set cover problem. The elements are $E = \{1,2,3,4,5\}$ and the subsets are $S_1 = \{2,3,4,5\}$, $S_2 = \{1,3,4,5\}$, $S_3 = \{1,2,4,5\}$, $S_4 = \{1,2,3,5\}$, and $S_5 = \{1,2,3,4\}$.

(a)

$$\begin{array}{rc} \min & x_1+x_2+x_2+x_3+x_4+x_5 \\ s.t. & x_2+x_3+x_4+x_5 \geq 1 \\ & x_1+x_3+x_4+x_5 \geq 1 \\ & x_1+x_2+x_4+x_5 \geq 1 \\ & x_1+x_2+x_3+x_5 \geq 1 \\ & x_1+x_2+x_3+x_4 \geq 1 \\ & x_1,x_2,x_3,x_4,x_5 \in \{0,1\} \end{array}$$

The optimal value is 2.

(b) LP-solution: $x_1 = x_2 = x_3 = x_4 = x_5 = 1/4$. The LP-value is $5/4$. (Other solutions are possible.)

(c)

$$\begin{array}{rc} \max & y_1+y_2+y_2+y_3+y_4+y_5 \\ s.t. & y_2+y_3+y_4+y_5 \leq 1 \\ & y_1+y_3+y_4+y_5 \leq 1 \\ & y_1+y_2+y_4+y_5 \leq 1 \\ & y_1+y_2+y_3+y_5 \leq 1 \\ & y_1+y_2+y_3+y_4 \leq 1 \\ & y_1,y_2,y_3,y_4,y_5 \geq 0. \end{array}$$

2. (a) Let $G = (V,E)$ be the graph, then the ILP is

$$\begin{array}{rcc} \min & \sum_{i\in V} x_i & \\ s.t. & x_i + x_j \geq 1 & \text{for all } (i,j) \in E. \\ & x_i \in \{0,1\} & \text{for all } i \in V. \end{array}$$

(b) Take the complete graph on n vertices. The minimum vertex cover has value $n-1$. The solution $x_i = 1/2$ for all $i \in V$ is feasible for the LP and has value $n/2$. So $\text{OPT}_{LP} \leq n/2$ and

$$\text{OPT}_{VC}/\text{OPT}_{LP} \geq (n-1)/(n/2) = 2 - 2/n.$$

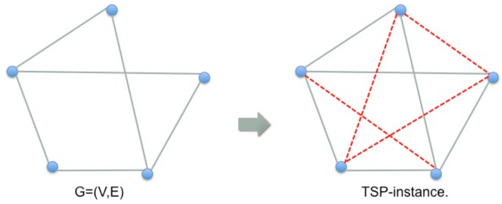
(c) Let $G = (V,E)$ be the graph, then the ILP is

$$\begin{array}{rcc} \max & \sum_{i\in V} x_i & \\ s.t. & x_i + x_j \leq 1 & \text{for all } (i,j) \in E. \\ & x_i \in \{0,1\} & \text{for all } i \in V. \end{array}$$

(d) Take the complete graph on n vertices. The maximum independent set has value 1. The solution $x_i = 1/2$ for all $i \in V$ is feasible for the LP and has value $n/2$. So $\text{OPT}_{LP} \geq n/2$ and

$$\text{OPT}_{LP}/\text{OPT}_{IS} \geq (n/2)/1 = n/2.$$

3. Reduce from the Hamiltonian Cycle problem. Given an instance $G = (V,E)$ of HC, form an instance of TSP by defining $c_{ij} = 1$ for all edges $(i, j) \in E$ and $c_{ij} = M$ for all $(i, j) \notin E$, where M is a large enough number. For example, $M = \alpha n$ is large enough. If there is no HC in G, then any TSP should use at least one of the edges of length M and the length of the optimal TSP tour is then at least $M+n-1 \geq \alpha n+n-1 > \alpha n$ (for $n \geq 2$).



Assume that there does exist some constant factor α-approximation algorithm. Let ALG be the value of the algorithm's solution when we apply it to the TSP instance. Then,

If G has a HC	$\Rightarrow$	$\text{OPT}_{TSP} = n \Rightarrow$	$\text{ALG} \leq \alpha n$.
If G has no HC	$\Rightarrow$	$\text{OPT}_{TSP} \geq M+n-1 > \alpha n \Rightarrow$	$\text{ALG} > \alpha n$.

We conclude that G has a HC if and only if $\text{ALG} \leq \alpha n$. Hence, we can use the algorithm to solve the HC problem. However, that problem is NP-complete. So no such α-approximation algorithm can exist.

4. (a)

$$
\begin{aligned}
(\text{LP}) \ \min \quad & Z = \sum_{j=1}^{n} w_j C_j \\
s.t. \quad & C_j \geqslant r_j + p_j & \text{for all jobs } j \\
& \sum_{j \in S} p_j C_j \geqslant \tfrac{1}{2} \left(\textstyle\sum_{j \in S} p_j\right)^2 & \text{for all sets } S \subseteq \{1, \ldots, n\}
\end{aligned}
$$

Remark 1: The constraint $C_j \geqslant 0$ is not needed since it is implied by the first constraint but it is OK to add the constraint.
Remark 2: In the second constraint one could add $\frac{1}{2}\sum_{j \in S} p_j^2$ on the right side.

(b) The optimum value is $2+3+4+5+6 = 20$.

(c) 5 (first constraint) $+2^5$ (second constraint) = 5+32 =37 constraints. (Better: It is 36 since $S = \emptyset$ gives no constraint.)

(d) next page:

(d)

The first constraint is clearly satisfied : $r_j + p_j = 2$ and $C_j > 2$ for all j.

For the second constraint we first order by completion time: $C_5 < C_4 < C_3 < C_3 < C_1$. We only need to verify it for 5 sets:

$$S = \{5\}: \qquad 2.8 \geq \frac{1}{2}(1)^2 \tag{1}$$

$$S = \{4,5\}: \qquad 2.9 + 2.8 \geq \frac{1}{2}(2)^2 \tag{2}$$

$$S = \{3,4,5\}: \qquad 3.0 + 2.9 + 2.8 \geq \frac{1}{2}(3)^2 \tag{3}$$

$$S = \{2,3,4,5\}: \qquad 3.1 + 3.0 + 2.9 + 2.8 \geq \frac{1}{2}(4)^2 \tag{4}$$

$$S = \{1,2,3,4,5\}: \qquad 3.2 + 3.1 + 3.0 + 2.9 + 2.8 \geq \frac{1}{2}(5)^2. \tag{5}$$

All these are satisfied so the solution is feasible. (If you included $\frac{1}{2}\sum_{j \in S} p_j^2$ in the second constraint of the LP then the solution is also feasible.

5. **(a)** Algorithm: Set each variable independently to True with probability $1/2$. For each clause, the probability that it is satisfied is: $1 - (1/2)^{l_j} \geq 1/2$, where l_j is the number of literals in the clause. If m is the number of clauses, then the expected number of clauses satisfied is at least $m/2 \geq \mathrm{OPT}/2$.

   (b) $3/4 + 3/4 + 3/4 + 3/4 + 7/8 = 3 + 7/8$

   (c) If n is the number of variables and m is the number of clauses, then the LP is

$$\begin{aligned}
\text{(LP)}\ \max \quad & Z = \sum_{j=1}^{m} w_j z_j & \\
s.t. \quad & \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geqslant z_j & \text{for all } j = 1 \dots m, \\
& 0 \leq y_i \leq 1 & \text{for all } i = 1 \dots n, \\
& 0 \leq z_j \leq 1 & \text{for all } j = 1 \dots m.
\end{aligned}$$

   Remark: The $y_i \leq 1$ can be removed since it is not needed.

6. **(a)**

$$\begin{aligned}
\text{(QP)}\ \max \quad & \tfrac{1}{2} \sum_{(i,j)} (1 - y_i y_j) w_{ij} & \\
s.t. \quad & y_i \in \{-1, 1\} & i = 1, \dots, n.
\end{aligned}$$

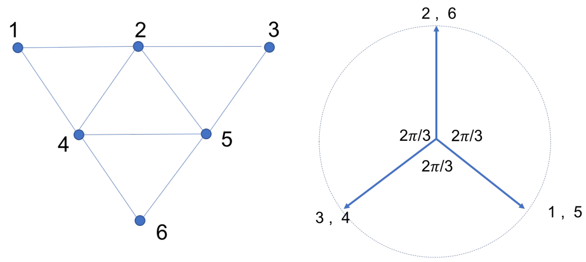
   Remark: It is also fine if you give the unweighted form ($\max \frac{1}{2} \sum_{(i,j) \in E} (1 - y_i y_j)$), or to write $y_i^2 = 1$ in stead of $y_i \in \{-1, 1\}$.

   (b)

$$\begin{aligned}
\text{(VP)}\ \max \quad & \tfrac{1}{2} \sum_{(i,j)} (1 - v_i \cdot v_j) w_{ij} & \\
s.t. \quad & v_i \cdot v_i = 1,\ v_i \in \mathbb{R}^n & i = 1, \dots, n.
\end{aligned}$$

(c) There is a cut of 6 edges and that is the maximum value since for each triangle at most two edges can be in the cut and there are 3 triangles that do not share any edges.

(d) Take 3 vectors with an angle of $2\pi/3$ between each pair. Assign each vertex to a vector such that the angle is $2\pi/3$ for any pair of vertices that are endpoint of an edge. (This can be done since the graph is 3-colorable.)



The value of this solution is $\frac{1}{2} \sum_{(i,j) \in E} (1 - v_i \cdot v_j) = \frac{1}{2} \cdot 9 \cdot (1 - -0.5) = 6.75$.