Name: Student number:

VU Amsterdam	Calculus 2 for BA (X_400636)
Faculty of Sciences	Exam 2
Dr. Senja Barthel	18-12-2023, 12:15-14:30 (+30 minutes extra time)

The use of a calculator, the book, or lecture notes is not permitted.
Do not just give answers, but write calculations and explain your steps.

You can score 33 points.

Question 1. (1+1+2+4+2 points)
Consider the function $f(x,y) = x^2 + xy$.

a) Compute the gradient of the function.

b) Compute the Hessian of the function.

c) Compute the directional derivative of the function in direction $v = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ at the point $p = (2,4)$.

d) Determine all maxima and minima of the function along the curve $\gamma(x) = 3x + 6$.

e) Sketch the gradient vector field of the function in the marked points $p_1 = (-1,2), p_2 = (0,-1), p_3 = (1,-1), p_4 = (1,1)$.

a) $\text{grad}_f(x,y) = \begin{bmatrix} 2x + y \\ x \end{bmatrix}$

b) $\text{Hess}_f(x,y) = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$

c) The directional derivative is
$\frac{v_1}{|v|} f_1(p_x, p_y) + \frac{v_2}{|v|} f_2(p_x, p_y) = \frac{4}{\sqrt{9+16}} 4 + 4 + \frac{-3}{\sqrt{9+16}} 2 = \frac{32}{5} - \frac{-6}{5} = \frac{26}{5}$.

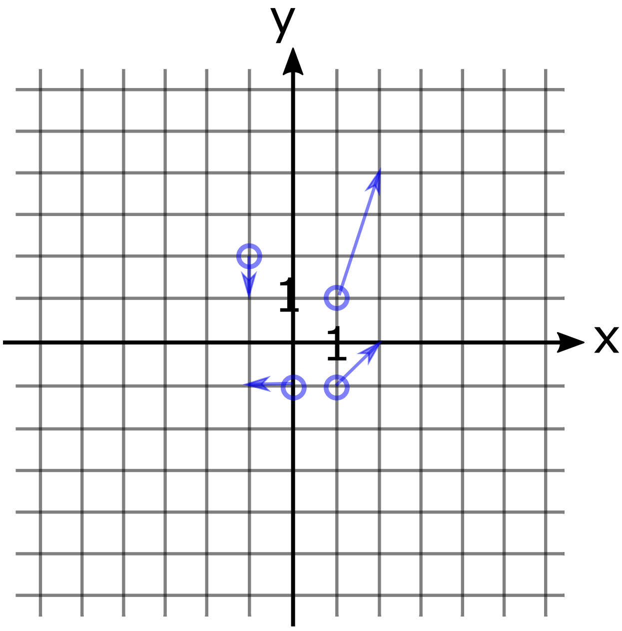
d) There are several possible approaches to this question.

First solution: Use Lagrange multiplier.
$f(x,y) = x^2 + xy$, $g(x,y) = 3x - y + 6$, $L(x,y,\lambda) = x^2 + xy + \lambda(3x - y + 6)$.
Find the critical point of $L(x,y,\lambda)$:

$$\left.\begin{aligned} 0 &= \tfrac{\partial L}{\partial x} = 2x + y + 3\lambda \\ 0 &= \tfrac{\partial L}{\partial y} = x - \lambda \Leftrightarrow x = \lambda \end{aligned}\right\} \Rightarrow y = -5x$$

$0 = \frac{\partial L}{\partial \lambda} = 3x - y + 6 \Rightarrow 8x = -6 \Rightarrow x = -\frac{3}{4}, y = \frac{15}{4}$.
Since in all points (and therefore also in the critical point) the first entry of the Hessian is $2 > 0$ and the determinant of the Hessian is $1 > 0$, the point is a minimum.

Second solution: Restrict the function to one variable by concatenation.
$f(x) = x^2 + x(3x+6) = x^2 + 3x^2 + 6x = 4x^2 + 6x$.
Find the critical points of $f(x)$:
$f'(x) = 8x + 6 = 0 \Rightarrow x = -\frac{3}{4}$. Therefore, $y = 3\frac{-3}{4} + 6 = -\frac{9}{4} + \frac{24}{4} = \frac{15}{4}$.
Since in all points (and therefore also in the critical point) the first entry of the Hessian is $2 > 0$ and the determinant of the Hessian is $1 > 0$, the point is a minimum.

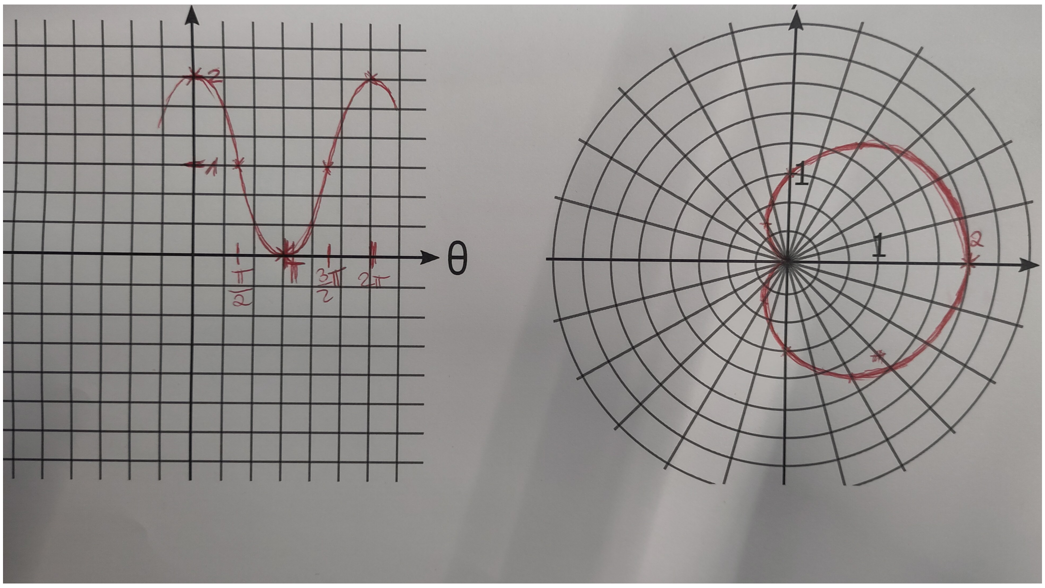
e) 

Question 2. (3+4 points)
Consider the polar curve $r = f(\theta) = 1 + \cos(\theta)$.

a) Rewrite the equation $r = 1+\cos(\theta)$ in Cartesian coordinates. (I.e., all r and θ should be replaced by x and y.) You do not need to simplify the solution.

b) Sketch the polar curve in both, the Polar and the Cartesian coordinate system. Do not forget to label the axes to indicate the step sizes you use.

a) With $r = \sqrt{x^2+y^2}$ and $x = r\cos(\theta)$, $x = r\cos(\theta) = \sqrt{x^2+y^2}\cos(\theta)$
$\Leftrightarrow \cos(\theta) = \frac{x}{\sqrt{x^2+y^2}}$, we have $r = 1 + \cos(\theta) \Leftrightarrow \sqrt{x^2+y^2} = 1 + \frac{x}{\sqrt{x^2+y^2}}$.
$\left(\Leftrightarrow 1 = \frac{1}{\sqrt{x^2+y^2}} + \frac{x}{x^2+y^2} \Leftrightarrow 1 = \frac{\sqrt{x^2+y^2}+x}{x^2+y^2} \right)$.

b) 

Question 3. (4 points)

Compute $\int_0^2 \int_0^y y^2 \mathrm{e}^{xy} dx\, dy$

$$\int_0^2 \int_0^y y^2 \mathrm{e}^{xy} dx\, dy = \int_0^2 y^2 \left[\frac{1}{y}\mathrm{e}^{xy}\right]_0^y dy = \int_0^2 y\left(\mathrm{e}^{y^2} - 1\right) dy = \left[\frac{\mathrm{e}^{y^2} - y^2}{2}\right]_0^2 = \frac{\mathrm{e}^4 - 5}{2}.$$

Question 4. (2 points)
What is the geometric meaning of the Jacobian determinant of a coordinate transformation (such as from Cartesian to polar coordinates or vice versa)?

The absolute value of the Jacobian is the ratio between corresponding area elements (in general: volume elements) in the different coordinate systems. Or: The Jacobian determinant gives the factor with which areas (in general: n-volumes) are stretched/shrunk by the coordinate transformation (in general: by a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$).

Question 5. (2+3 points)

a) Compute the real part and the imaginary part of $(a + ib)^2$.

b) Explain how real numbers relate to their complex conjugates.

a) $(a + ib)^2 = a^2 + 2aib + (ib)^2 = a^2 - b^2 + i2ab$. Therefore, the real part of $(a + ib)^2$ is $a^2 - b^2$ and the imaginary part is $2ab$.

b) The complex conjugate of the complex number $a + ib$ is $a - ib$. Since real numbers are exactly the complex numbers with $b = 0$, real numbers are their own complex conjugates.

Question 6. (1+4 points)

a) State the general form of a linear ordinary nonhomogeneous differential equation of order n.

b) Let $y_1(x)$ be a solution to a linear ordinary nonhomogeneous differential equation of order n, and let $y_0(x)$ be a solution to the corresponding homogeneous ordinary differential equation.
Show that the function $y(x) = y_1(x) + y_0(x)$ is also a solution of the same linear nonhomogeneous differential equation.

a) $a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y^{(1)}(x) + a_0(x)y(x) = f(x)$.

b) y_1 being a solution to the nonhomogeneous differential equation means that

$$a_n(x)y_1^{(n)}(x) + a_{n-1}(x)y_1^{(n-1)}(x) + \cdots + a_1(x)y_1^{(1)}(x) + a_0(x)y_1(x) = f(x).$$

y_0 being a solution to the homogeneous corresponding differential equation means that

$$a_n(x)y_0^{(n)}(x) + a_{n-1}(x)y_0^{(n-1)}(x) + \cdots + a_1(x)y_0^{(1)}(x) + a_0(x)y_0(x) = 0.$$

Adding the two equation gives

$$\begin{aligned}
f(x) =& f(x) + 0 \\
=& a_n(x)y_1^{(n)}(x) + a_{n-1}(x)y_1^{(n-1)}(x) + \cdots + a_1(x)y_1^{(1)}(x) + a_0(x)y_1(x) \\
&+ a_n(x)y_0^{(n)}(x) + a_{n-1}(x)y_0^{(n-1)}(x) + \cdots + a_1(x)y_0^{(1)}(x) + a_0(x)y_0(x) \\
=& a_n(x)(y_1^{(n)}(x) + y_0^{(n)}(x)) + a_{n-1}(x)(y_1^{(n-1)}(x) + y_0^{(n-1)}(x)) + \dots \\
&\cdots + a_1(x)(y_1^{(1)}(x) + y_0^{(1)}(x)) + a_0(x)(y_1(x) + y_0(x)) \\
=& a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y^{(1)}(x) + a_0(x)y(x)
\end{aligned}$$

which shows the claim.

End of exam.