VU Amsterdam	Calculus 2 for BA (X_400636)
Faculty of Sciences	Exam 1
Dr. Senja Barthel	22-11-2023, 18:45-21:00
	(+30 minutes extra time)

The use of a calculator, the book, or lecture notes is not permitted.
Do not just give answers, but write calculations and explain your steps.

You can score 36 points.

Question 1. (2 points)

Define what it means for a sequence $\{a_n\}$ to converge towards a limit L.

The sequence $\{a_n\}$ converges to the limit L if for every positive real number ϵ, there exists an integer N such that $|a_n - L| < \epsilon$ for all $n \geq N$.

Question 2. (4 points)

Consider the sequence

$$a_1 = 1,\ a_2 = -2,\ a_3 = 1,\ a_n = -a_{n-3} - a_{n-2} - a_{n-1} \text{ for all } n > 3.$$

Determine whether this sequence is

a) increasing, decreasing, alternating, or none of the previous,

b) bounded (above and/or below),

c) convergent or divergent.

Writing out the elements gives

$$1, -2, 1, 0, 1, -2, 1, 0, 1, -2, 1, 0, \ldots$$

a) and therefore not increasing ($a_2 < a_1$), not decreasing ($a_3 > a_2$), and not alternating ($a_2 \cdot a_3$ is not strictly negative).

b) The sequence is bounded above by 1 and bounded below by -2.

c) The sequence is repeating four non-constant values, and therefore divergent.

Question 3. (5 points)

Find the series representation on an interval including $x = 0$ for the function

$$f(x) = \frac{1+x^3}{1+x^2}.$$

For what values of x is the representation valid?

$\frac{1+x^3}{1+x^2} = (1+x^3)\frac{1}{1-(-x^2)}$. Substituting $-x^2$ in the geometric series gives for $|x| < 1$:
$(1+x^3)\frac{1}{1-(-x^2)} = (1+x^3)\sum_{t=0}^{\infty}(-x^2)^t = (1+x^3)\sum_{t=0}^{\infty}(-1)^t x^{2t}$

$= (1+x^3)(1-x^2+x^4-x^6+x^8-\dots) = 1-x^2+x^3+x^4-x^5-x^6+x^7+x^8-\dots$
$= 1-x^2+\sum_{n=2}^{\infty}(-1)^n(x^{2n-1}+x^{2n})$.
The representation is valid for $|x| < 1$ and no other values of x, since this particular substitution does not change the convergence interval, and the geometric series diverges for all values outside the interval. (It is also correct to show that the resulting series diverges for $x \leq -1, x \geq 1$ directly.)

Question 4. (6 points)

(+1 point for each correct answer, -1 point for each wrong answer.
If the sum of all points is negative, the question is graded with zero points.)

Tick all true statements. Read carefully.

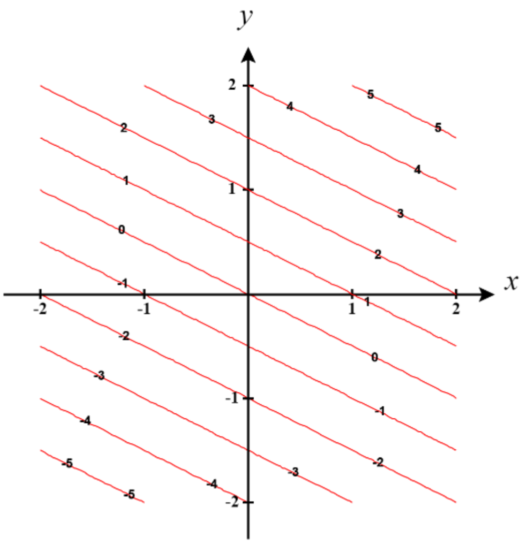
- ◯ Every conditionally convergent series is alternating.
  false
- ◯ Every alternating series is conditionally convergent.
  false
- ◯ There exists an alternating series that is absolutely convergent.
  true
- ◯ Reordering a conditionally convergent series can make it absolutely convergent.
  false
- ◯ Every convergent negative series is absolutely convergent.
  true
- ◯ Every convergent positive series is absolutely convergent.
  true

Question 5. (3+2 points)

Consider the function $f(x, y) = x + 2y$.

a) Make a contour plot for the function by drawing the seven isolevels for the values $-3, -2, -1, 0, 1, 2, 3$.

b) Give a formula describing the geometric object that is defined by intersecting the graph of $f(x, y)$ with the xz-plane. What kind of object is this?

a) 

b) This is a line with equation $z = x$.

Question 6. (1+1+2 points)

a) Give the definition of the first partial derivative $f_1(x, y)$ of a differentiable function $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$.

b) Describe the geometric meaning of the first derivative $f_1(a, b)$ taken in the point (a, b).

c) Compute the partial derivatives of the function $f(x, y) = 3x^2\sqrt{y}$.

a) The first partial derivative of the function $f(x)$ is defined by

$$f_1(x, y) := \lim_{h\to\infty} \frac{f(x + h, b) - f(x, y)}{h}.$$

b) The partial derivative $f_1(a, b)$ measures the rate of change of $f(x, y)$ in (a, b) in $x-$direction.

c) $f_1(x, y) = 6x\sqrt{y}$, $f_2(x, y) = \frac{3}{2}\frac{x^2}{\sqrt{y}}$.

Question 7. (5 points)

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be three vectors in $\mathbb{R}^3$ and t be a real value. Verify the identity

$$\boldsymbol{u} \bullet (\boldsymbol{v} \times t\boldsymbol{w}) = t\boldsymbol{v} \bullet (\boldsymbol{w} \times \boldsymbol{u}).$$

The left side written out becomes $\boldsymbol{u} \bullet (\boldsymbol{v} \times t\boldsymbol{w}) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \bullet \left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times t \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right)$

$= u_1(v_2tw_3 - v_3tw_2) + u_2(v_3tw_1 - v_1tw_3) + u_3(v_1tw_2 - v_2tw_1)$

$= t(u_1 v_2 w_3 - u_1 v_3 w_2 + u_2 v_3 w_1 - u_2 v_1 w_3 + u_3 v_1 w_2 - u_3 v_2 w_1)$
The right side written out becomes $t\boldsymbol{v} \bullet (\boldsymbol{w} \times \boldsymbol{u}) = t \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \bullet \left(\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \times \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \right)$
$= tv_1(w_2 u_3 - w_3 u_2) + tv_2(w_3 u_1 - w_1 u_3) + tv_3(w_1 u_2 - w_2 u_1)$
$= t(v_1 w_2 u_3 - v_1 w_3 u_2 + v_2 w_3 u_1 - v_2 w_1 u_3 + v_3 w_1 u_2 - v_3 w_2 u_1)$
Comparing the two results term-wise shows that they are indeed equal.

Question 8. (5 points)

Find the equation of the plane that

a) contains the line of intersection of the two planes $2x+3y-z = 0$ and $x-4y+2z = -5$,

b) and passes through the point with coordinates $(-2, 0, -1)$.

All planes, except for the second plane, that contain the line of intersection between the two given planes are given by an equation of the form

$$(2x + 3y - z) + \lambda(x - 4y + 2z + 5) = 0, \lambda \in \mathbb{R}.$$

Since $-2 - 4 \cdot 0 + 2 \cdot (-1) = -2 - 2 = -4 \neq -5)$, the point with coordinates (-2,0,-1) does not lie on the second plane, and we can find a value for λ such that the equation above describes the wanted plane. To find λ, the coordinates (-2,0,-1) are chosen for the x,y,z values in the equation, and we obtain

$2 \cdot (-2) + 3 \cdot 0 - (-1) + \lambda(-2 - 4 \cdot 0 + 2 \cdot (-1) + 5) = -4 + 1 + \lambda(-2 - 2 + 5) = -3 + \lambda = 0.$

Therefore, $\lambda = 3$ and the equation of the plane is

$(2x+3y-z)+3(x-4y+2z+5) = 0 \Leftrightarrow 2x+3y-z+3x-12y+6z+15 = 0 \Leftrightarrow 5x-9y+5z = -15.$

End of exam.