VU Amsterdam	Calculus 2 for BA (X_400636)
Faculty of Sciences	Practice questions for Exam 2
Dr. Senja Barthel	Autumnn 2022

The use of a calculator, the book, or lecture notes is <u>not</u> permitted.
Do not just give answers, but write calculations and explain your steps.

Question 1: (3 points)
The theorem of Schwarz states that if

(a) two mixed n-th order partial derivatives of a function f involve the same differentiations but in different order

(b) and if all those partials are continuous at a point P

(c) and all partials of f of order less than n are continuous in a neighbourhood of P

Then the two partials are equal at the point P.

Explain how the proof of Schwarz' theorem uses (c).

The proof uses the Mean-value theorem iteratively on partial derivatives to find a point in a tiny rectangle with corner P at which the mixed partials are equal. The Mean-value theorem can only be applied to continuous functions, therefore the partial derivatives all have to be continuous, on the tiny rectangle at least (which is ensured if they are continuous in a neighbourhood of P that then will contain such rectangle), for the proof to work.[1]

Question 2: (5 points)
Let $f(x, y) = x + \sin x$ be a function from R^2 to R.

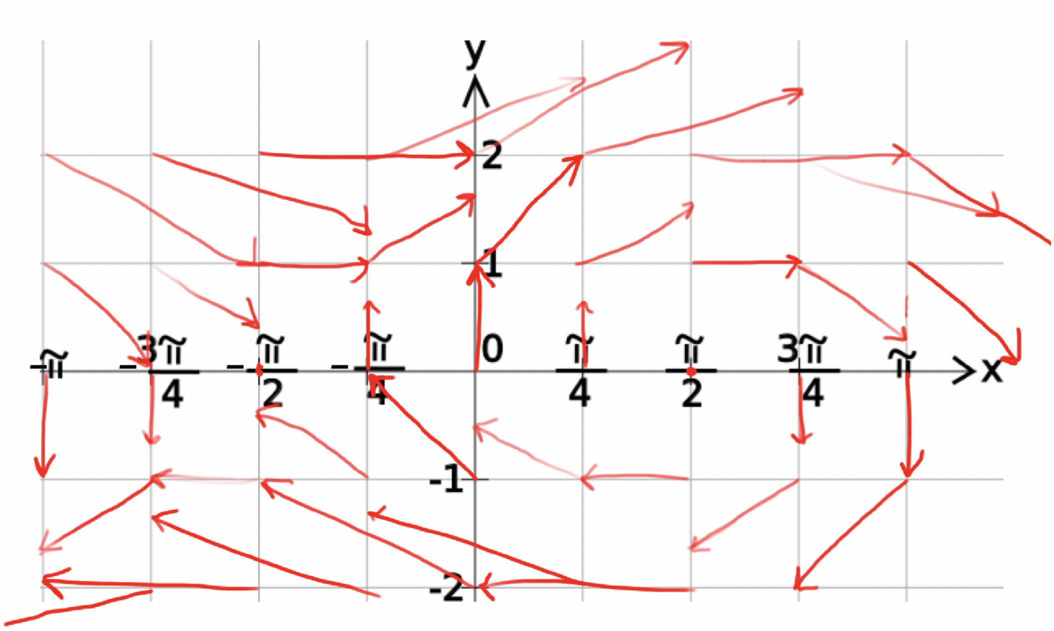
(a) Compute the gradient $\mathbf{grad}_f(a, b)$ of f in the point (a, b).
$\frac{\partial f}{\partial x} = 1 + \cos x$. Since the function does not depend on y, the partial derivative in y-direction is zero everywhere. Therefore, $\mathbf{grad}_f(a, b) = \begin{bmatrix} 1 + \cos a \\ 0 \end{bmatrix}$

(b) Compute the directional derivative of $f(x, y)$ in the point with coordinates $(-\frac{\pi}{3}, 0.2)$, in direction of the vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.
The length of the vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is 5. Therefore, the directional derivative is
$\frac{3}{5}(1 + \cos(x)) + \frac{4}{5} \cdot 0 = \frac{3}{5}(1 + \cos(x))$.
In the point $(-\frac{\pi}{3}, 0.2)$ the directional derivative is $\frac{3}{5}(1 + \cos(-\frac{\pi}{3})) = \frac{6}{10} + \frac{3}{5} \cdot \frac{1}{2} = \frac{9}{10}$.

Question 3: (4 points)
The gradient of a function is given as the vector field $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ \cos(x) \end{bmatrix}$.
Draw the vector field by attaching a vector to each of the grid points below:

[1] The statement of the theorem is also not true in general if the requirement of continuity is dropped, so this is not specific to the proof that we saw in the lecture. But this was not asked here

Solution:



Question 4: (8 points)
Determine all points where the function $f : R^2 \to R$,
$f(x, y) = (x-1)^3 + 3(x-1)^2 - xy^2 - 2$ has a maximum or minimum.

Necessary condition for the existence of an extremum: $\text{grad}_{(x,y)} f = 0$.

$$\frac{\partial f}{\partial x}(x, y) = 3(x-1)^2 + 6(x-1) - y^2 = 0$$
$$\frac{\partial f}{\partial y}(x, y) = -2xy = 0$$

From the second equation: $x = 0$ or $y = 0$.
For $x = 0$ it follows from the first equation that $3 - 6 - y^2 = 0 \Leftrightarrow y^2 = -3$. This equation has no real solution, therefore the function has no critical values with $x = 0$.
For $y = 0$ it follows from the first equation that $3(x-1)^2 + 6(x-1) = 0 \Leftrightarrow x^2 = 1$.
The critical values of f are therefore $(-1, 0)$ and $(1, 0)$.
The Hessian matrix of f is

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x \partial x}(x, y) & \frac{\partial^2 f}{\partial x \partial y}(x, y) \\ \frac{\partial^2 f}{\partial x \partial y}(x, y) & \frac{\partial^2 f}{\partial y \partial y}(x, y) \end{bmatrix} = \begin{bmatrix} 6x & -2y \\ -2y & -2x \end{bmatrix}$$

The determinant of the Hessian is $-12x^2 < 0$ for both critical points. Therefore no local extrema exist.
Since the function is defined on R, no boundary points have to be considered. (If the domain had boundary, we would know that the maximum and minimum are taken on the boundary ad we would have to find them.)

Question 5: (5 points)
State for each of the following five statements whether they are true or false (here you do not need to explain your answer).
+1 point for each correct answer, -1 point for each wrong answer.
If the sum of points is negative, the question is graded with 0 points.

a) If all partial derivatives of a function exist in a point p, then all directional derivatives exist in p as well.
false

b) If all directional derivatives of a function exist in a point p, then all partial derivatives exist in p as well.
true

c) If a function is differentiable in a point p, then all partial derivatives exist in p.
true

d) If a function is differentiable in a point p, then all directional derivatives exist in p.
true

e) If all directional derivatives of a function exist in a point p, then the function is differentiable in p.
false

Question 6: (10 points)
Compute the volume of the object whose base is the region in the xy-plane that is bounded by the three curves $y = \mathrm{e}^x$, $y = \mathrm{e}$, $x = 0$, and the cross sections of the object parallel to the xz-plane are squares.
(Hint: It might help to sketch the object.)

We want to integrate the area of the squares along the y-axis. The intersection of $y = \mathrm{e}^x$ with the y-axis is at $\mathrm{e}^{x=0}$, therefore $y = 1$. The upper bound of the integral is $y = \mathrm{e}$. The squares have sides of lengths $\ln(y)$, since ln is the inverse of the exponential function.
The volume of the object is therefore given by

$$V = \int_1^{\mathrm{e}} (\ln y)^2 \, dy$$

The integral can be solved using integration by parts , $u = (\ln y)^2$, $v = 1$:

$$\begin{aligned}\int_1^{\mathrm{e}} (\ln y)^2 \, dy &= \left[y \ln^2(y)\right]_1^{\mathrm{e}} - \int_1^{\mathrm{e}} y \frac{2 \ln y}{y} \, dy = \left[y \ln^2(y)\right]_1^{\mathrm{e}} - 2 \int_1^{\mathrm{e}} \ln y \, dy \\ &= \left[y \ln^2(y) - 2(y(\ln y - 1))\right]_1^{\mathrm{e}}) = \left[y(\ln^2 y - 2 \ln y + 2)\right]_1^{\mathrm{e}} \\ &= \mathrm{e} - 2\end{aligned}$$

Question 7: (6 points)

a) Argue that $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$ for $n \in \mathbb{N}$, $-\pi < \theta \leq \pi$.
Multiplying two complex numbers is achieved by multiplying their moduli and adding their arguments, i.e. for $z = |z|(\cos(\theta) + i\sin(\theta))$ and $w = |w|(\cos(\phi) + i\sin(\phi))$, $z + w = |zw|(\cos(\theta + \phi) + i\sin(\theta + \phi)).(\cos(\theta) + i\sin(\theta))$
Since complex number $z = \cos(\theta) + i\sin(\theta)$ has modulus $|z| = \cos^2(\theta) + \sin^2(\theta) = 1$ and argument $\arg z = \theta$, it follows that $z^n = 1^n(\cos(n\theta) + i\sin(n\theta)) = (\cos(n\theta) + i\sin(n\theta))$

b) Let $z = 2 + 2i$, compute z^3.
$(2 + 2i)(2 + 2i)(2 + 2i) = 8i(2 + 2i) = -16 + 16i$

Question 8: (8 points)
Show that $y = x$ is a solution of $x^2y'' + 21xy' - 2y = 0$ on the interval $(0, \infty)$, and find the general solution on this interval.

If $y_1 = x$ on $(0, \infty)$, then $x^2y_1'' + 2xy_1' - 2y_1 = 0 + 2x - 2x = 0$, which shows that y_1 is a solution of $x^2y'' + 21xy' - 2y = 0$.
Let $y = xv(x)$. Then $y' = xv' + v$ and $y'' = xv'' + 2v'$ and the differential equation becomes $x^2y'' + 21xy' - 2y = x^3v'' + 2x^2v' + 2x^2v' + 2xv - 2xv = x^2(xv'' + 4v')$.
It follows that y satisfies $x^2y'' + 21xy' - 2y = 0$ if $w = v'$ satisfies $xw' + 4w = 0$.
Separating variables shows that this equation has solution $v' = w = -3C_1x^{-4}$, therefore $v = C1x^{-3} + C_2$. It follows that $x^2y'' + 21xy' - 2y = 0$ has solution $y = xv = C_1x^{-2} + C_2x$.

End of exam.