| VU Amsterdam | Calculus 2 for BA (X_400636) |
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| Faculty of Sciences | Exam 2 |
| Dr. Senja Barthel | 19-12-2022, 12:15-14:30<br>(+30 minutes extra time) |

**The use of a calculator, the book, or lecture notes is <u>not</u> permitted.**
**Do not just give answers, but write calculations and explain your steps.**

**You can score 51 points.**

**Question 1.** (1+3+2 points)

a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function for which all second-order derivatives exist. Give the general form of the Hessian matrix of the function $f$.

b) Compute the Hessian matrix of the function $f(x, y) = yx\cos(2x) + 15\sin(y)$.

c) State a theorem that explains why two entries of the Hessian matrix in b) are equal.

**Question 2.** (2+2+2+2+4 points)
For the function $F(x, y) = -\sin(x) - 0.5y$

a) compute the gradient of $F(x, y)$.

b) describe the geometric meaning of the gradient $\mathbf{grad}f(a, b)$ of $f$ in the point $(a, b)$.

c) compute the partial derivatives in the point $(x, y) = (0, -\pi)$.

d) give the directional derivative $D_{\mathbf{v}}F(x, y) = \frac{\partial F}{\partial v}(x, y)$ for $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ with $\sqrt{a^2 + b^2} = 13$.

e) draw the gradient vector field in all grid points of Figure 1.

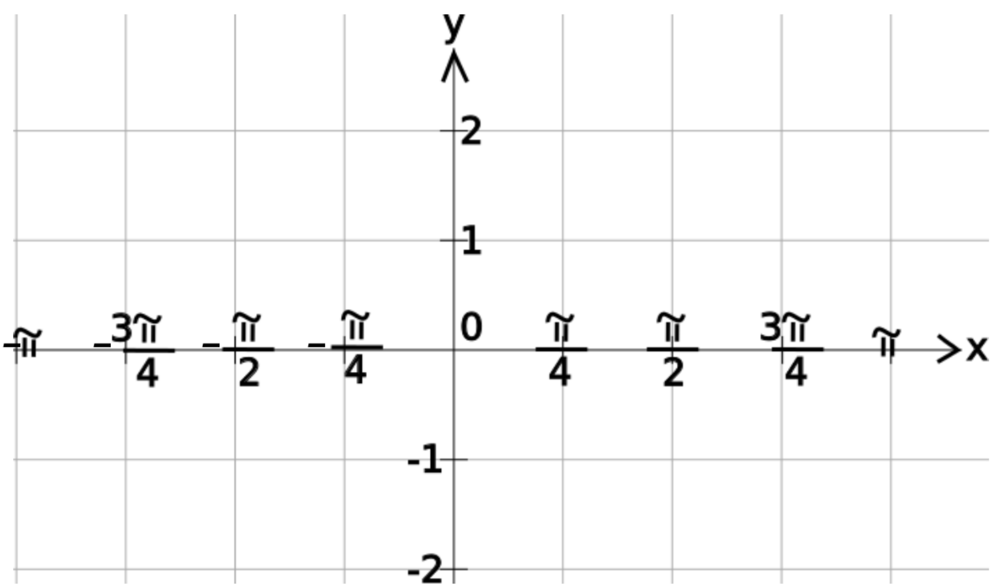


Figure 1: The figure for Question 5e. You can draw the gradient vector field in here or work on your own paper.

**Exam continues on next page.**

**Question 3.** (9 points)
For the function $f(x, y) = x + 2y$ defined on the disk $x^2 + y^2 \leq 1$

a) test the function for critical points on the interior of the disk $\{(x, y)|x^2 + y^2 < 1\}$.

b) determine whether the function has maxima and minima on the boundary of the disc $\{(x, y)|x^2 + y^2 = 1\}$.
Hint: If $\phi$ is an angle such that $\tan(\phi) = 2$, then $\cos(\phi) = \frac{1}{\sqrt{5}}$ or $\cos(\phi) = -\frac{1}{\sqrt{5}}$ and $\sin(\phi) = \frac{2}{\sqrt{5}}$ or $\sin(\phi) = -\frac{2}{\sqrt{5}}$.

**Question 4.** (6 points)
Integrate the function $f(x, y) = xy^2$ over the area that is enclosed by the parabola $y = x^2$ and the line $y = 2x$. (You do not need to simplify the final result.)

**Question 5.** (4 points)
State for each of the following four statements whether they are true or false (here you do not need to explain your answer).
+1 point for each correct answer, -1 point for each wrong answer.
If the sum of points is negative, the question is graded with 0 points.

For $f, g : D \to \mathbb{R}$ be continuous functions, $D \subset \mathbb{R}^2$:

a) $\left|\int\int_D f(x, y)dA\right| \leq \int\int_D |f(x, y)|dA$

b) For $D_1 \subset D_2 \subset D$: $\int\int_{D_1} f(x, y)dA \leq \int\int_{D_2} f(x, y)dA$

c) $\int\int_D f(x, y)dA - \int\int_D g(x, y)dA = \int\int_D (f - g)(x, y)dA$

d) $\int\int_{D_1} |f(x, y)|dA + \int\int_{D_2} |f(x, y)|dA \geq \int\int_{D_1 \cup D_2} |f(x, y)|dA$

**Question 6.** (4 points)
Write the complex number $\frac{i^5 + 5}{(1 + i)^5}$ in the form $x + iy$ with $x, y \in \mathbb{R}$.

**Question 7.** (2+3 points)

a) Describe the $y$-axis of the Cartesian coordinate system of $\mathbb{R}^2$ in polar coordinates. That is, give the polar coordinates of the points with Cartesian coordinates $(0, y)$.

b) The following coordinates are given in Cartesian coordinates: $p_1 = (0, 5)$, $p_2 = (7, 0)$, $p_3 = (2, 2)$. Transform them into polar coordinates.

**Question 8.** (5 points)
Determine the solution of the differential equation $\begin{cases} \frac{dy}{dx} = x^2y^2 + y^2 \\ y(3) = 1 \end{cases}$.

**End of exam.**