VU Amsterdam	Calculus 2 for BA (X_400636)
Faculty of Sciences	Exam 2
Dr. Senja Barthel	19-12-2022, 12:15-14:30 (+30 minutes extra time)

The use of a calculator, the book, or lecture notes is not permitted.
Do not just give answers, but write calculations and explain your steps.

You can score 51 points.

Question 1. (1+3+2 points)

a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function for which all second-order derivatives exist. Give the general form of the Hessian matrix of the function f.

b) Compute the Hessian matrix of the function $f(x, y) = yx\cos(2x) + 15\sin(y)$.

c) State a theorem that explains why two entries of the Hessian matrix in b) are equal.

a) $\begin{bmatrix} \frac{\partial^2 f}{\partial x \partial x} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial x \partial y} \end{bmatrix}$

b) Since

$\frac{\partial}{\partial x} f(x, y) = y(\cos(2x) - 2x\sin(2x))$,
$\frac{\partial^2}{\partial x \partial x} f(x, y) = -4y(x\cos(2x) + \sin(2x))$,
$\frac{\partial}{\partial y \partial x} f(x, y) = \cos(2x) - 2x\sin(2x)$,
$\frac{\partial}{\partial y} f(x, y) = x\cos(2x) + 15\cos(y)$,
$\frac{\partial}{\partial x \partial y} f(x, y) = \cos(2x) - 2x\sin(2x)$,
$\frac{\partial^2}{\partial y \partial y} f(x, y) = -15\sin(y)$

$$\text{Hess}_f(x, y) = \begin{bmatrix} -4y(x\cos(2x) + \sin(2x)) & \cos(2x) - 2x\sin(2x) \\ \cos(2x) - 2x\sin(2x) & -15\sin(y) \end{bmatrix}$$

c) The two mixed derivatives are equal by the Theorem of Schwarz. The theorem states that the order of differentiation in an n-order partial derivative is irrelevant for the result, as long as all $(n-1)$-order partial derivatives are continuous. The assumption on smoothness is The conditions of the theorem are fulfilled since all partial derivatives are continuous as concatenation of continuous functions.

Question 2. (2+2+2+2+4 points)
For the function $F(x, y) = -\sin(x) - 0.5y$

a) compute the gradient of $F(x, y)$.

b) describe the geometric meaning of the gradient $\mathbf{grad} f(a, b)$ of f in the point (a, b).

c) compute the partial derivatives in the point $(x, y) = (0, -\pi)$.

d) give the directional derivative $D_{\mathbf{v}} F(x, y) = \frac{\partial F}{\partial v}(x, y)$ for $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ with $\sqrt{a^2 + b^2} = 13$.

e) draw the gradient vector field in all grid points of Figure 1.

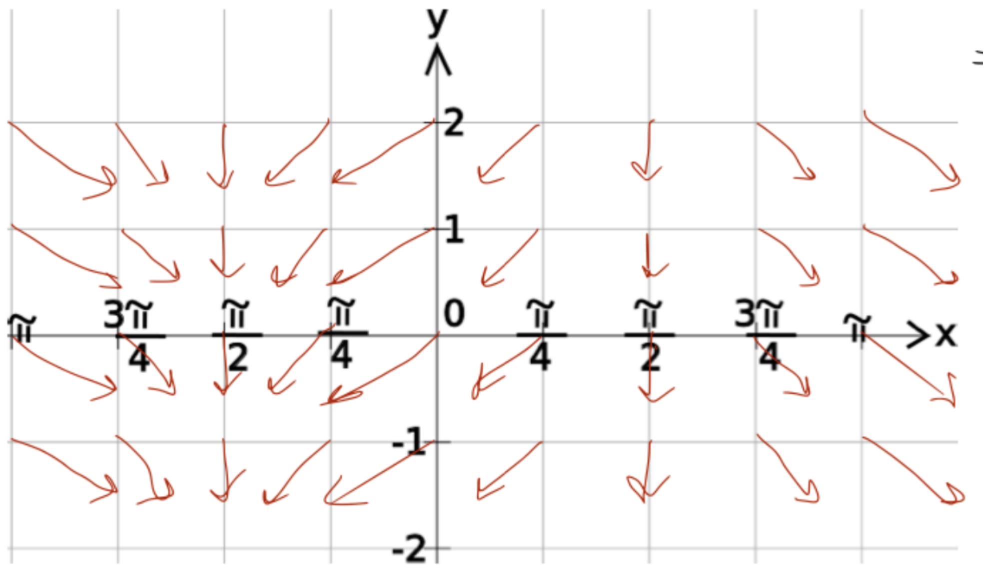
a) $\text{grad}_f(x,y) = \begin{pmatrix} -\cos x \\ -0.5 \end{pmatrix}$

b) The gradient points into the direction of steepest ascent of f in the point (a,b). Its lengths is the steepness, i.e. the directional derivative $\frac{\partial f}{\partial v}(a,b)$, where v is the direction of steepest ascent (the direction given by the gradient itself).

c) $\frac{\partial F}{\partial x}(x,y) = -\cos x$, $\frac{\partial F}{\partial y}(x,y) = -0.5y$.
In the point $(x,y) = (0,-\pi)$ this is $\frac{\partial F}{\partial x}(0,-\pi) = -1$, $\frac{\partial F}{\partial y}(0,-\pi) = -0.5$

d) $-\frac{a}{13}\cos x - \frac{b}{13}0.5 = -\frac{a}{13}\cos x - \frac{b}{26}$

e) the gradient vector field:



Question 3. (9 points)
For the function $f(x,y) = x + 2y$ defined on the disk $x^2 + y^2 \leq 1$

a) test the function for critical points on the interior of the disk $\{(x,y)|x^2 + y^2 < 1\}$.

b) determine whether the function has maxima and minima on the boundary of the disc $\{(x,y)|x^2 + y^2 = 1\}$.
Hint: If ϕ is an angle such that $\tan(\phi) = 2$, then $\cos(\phi) = \frac{1}{\sqrt{5}}$ or $\cos(\phi) = -\frac{1}{\sqrt{5}}$ and $\sin(\phi) = \frac{2}{\sqrt{5}}$ or $\sin(\phi) = -\frac{2}{\sqrt{5}}$.

a) The function has no critical points for (x,y) with $x^2 + y^2 < 1$, since $f_1 = 1$ and $f_2 = 2$.

b) Since f is continuous on a closed and bounded set in the plane, the maximum and minimum values of f must exist. Since they are not taken in the interior of the disc, they must be taken on the boundary, that is, on points of the circle $x^2+y^2 = 1$. This circle can be parametrised as $x = \cos t$, $y = \sin t$, which gives for $f(x,y)$ restricted to the circle $f(\cos t, \sin t) = \cos t + 2\sin t = g(t)$. Checking for critical points of g: $0 = g'(t) = -\sin t + 2\cos t$. Thus $\tan t = 2$, and $x = \pm\frac{1}{\sqrt{5}}$, $y = \pm\frac{2}{\sqrt{5}}$. For the tangent to be positive, both x and y must have the same sign. Therefore, the critical points are $(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}})$, where f has value $-\sqrt{5}$ (1 point), and $(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$, where f has value $\sqrt{5}$. Thus the maximum and minimum values of $f(x, y)$ on the disk $x^2 + y^2 \leq 1$ are $-\sqrt{5}$ and $-\sqrt{5}$ respectively.

Question 4. (6 points)
Integrate the function $f(x, y) = xy^2$ over the area that is enclosed by the parabola $y = x^2$ and the line $y = 2x$. (You do not need to simplify the final result.)

One needs to compute the integral $\int_0^2 \int_{x^2}^{2x} xy^2 \, dy \, dx$.

$$\int_0^2 \int_{x^2}^{2x} xy^2 \, dy \, dx = \int_0^2 x\frac{1}{3} \left[y^3\right]_{x^2}^{2x} \, dx = \int_0^2 x\frac{1}{3} \left((2x)^3 - x^6\right) \, dx$$
$$= \int_0^2 \frac{8}{3}x^4 - \frac{x^7}{3} \, dx = \left[\frac{8}{15}x^5 - \frac{1}{24}x^8\right]_0^2 = \frac{8}{15}2^5 - \frac{1}{24}2^8 \left(= \frac{32}{5}\right).$$

Question 5. (4 points)
State for each of the following four statements whether they are true or false (here you do not need to explain your answer).
+1 point for each correct answer, -1 point for each wrong answer.
If the sum of points is negative, the question is graded with 0 points.

For $f, g : D \to \mathbb{R}$ be continuous functions, $D \subset \mathbb{R}^2$:

a) $\left|\int\int_D f(x, y)dA\right| \leq \int\int_D |f(x, y)|dA$ — true

b) For $D_1 \subset D_2 \subset D$: $\int\int_{D_1} f(x, y)dA \leq \int\int_{D_2} f(x, y)dA$ — false

c) $\int\int_D f(x, y)dA - \int\int_D g(x, y)dA = \int\int_D (f - g)(x, y)dA$ — true

d) $\int\int_{D_1} |f(x, y)|dA + \int\int_{D_2} |f(x, y)|dA \geq \int\int_{D_1 \cup D_2} |f(x, y)|dA$ — true

Question 6. (4 points)
Write the complex number $\frac{i^5 + 5}{(1 + i)^5}$ in the form $x + iy$ with $x, y \in \mathbb{R}$.

$$\frac{i^5 + 5}{(1 + i)^5} = \frac{i + 5}{(1 + 2i + i^2)(1 + 2i + i^2)(1 + i)} = \frac{i + 5}{(1 + 2i - 1)(1 + 2i - 1)(1 + i)}$$
$$\frac{i + 5}{-4(1 + i)} = \frac{i + 5}{-4 - 4i} = \frac{i + 5}{-4 - 4i} = -\frac{3}{4} + \frac{1}{2}i$$

Question 7. (2+3 points)

a) Describe the y-axis of the Cartesian coordinate system of $\mathbb{R}^2$ in polar coordinates. That is, give the polar coordinates of the points with Cartesian coordinates $(0, y)$.

b) The following coordinates are given in Cartesian coordinates: $p_1 = (0, 5)$, $p_2 = (7, 0)$, $p_3 = (2, 2)$. Transform them into polar coordinates.

a) These are the two half lines with coordinates $(r, -\frac{\pi}{2})$ and $(r, \frac{\pi}{2})$ with $r \in [0, \infty[$, for the angle between $]-\pi, \pi]$. (Or $(r, \frac{\pi}{2})$ and $(r, \frac{3\pi}{2})$ with $r \in [0, \infty[$, for the angle between $[0, 2\pi[$)

b) Coordinates in polar coordinates:
$p_1 = [5, \frac{\pi}{2}]$, $p_2 = [7, 0]$, $p_3 = [\sqrt{8}, \frac{\pi}{4}]$

Question 8. (5 points)

Determine the solution of the differential equation $\begin{cases} \frac{dy}{dx} = x^2y^2 + y^2 \\ y(3) = 1 \end{cases}$.

Separating variables: $\frac{dy}{dx} = x^2y^2 + y^2 = y^2(1 + x^2)$, therefore $\frac{1}{y^2}dy = (1 + x^2)dx$.
Integrating both sides: $-\frac{1}{y} = x + \frac{1}{3}x^3 + C$.
Find the constant by using $y(3) = 1$: $-1 = 3 + \frac{1}{3}3^3 + C \Leftrightarrow -C = 12 + 1 = 13 \Leftrightarrow C = -13$.
Therefore the solution is $y(x) = -\frac{1}{x + \frac{1}{3}x^3 - 13}$.

End of exam.