VU Amsterdam	Calculus 2 for BA (X_400636)
Faculty of Sciences	Practice Exam 1
Dr. Senja Barthel	Autumnn 2022, 135 minutes (+30 minutes extra time)

The use of a calculator, the book, or lecture notes is not permitted.
Only the sheet 'Flow chart series convergence' may be used.
Do not just give answers, but write calculations and explain your steps.

You can score 57 points.

Question 1. (2+2+2+2 points)

For the following statements, decide whether they are true or false. If a statement is true, give an argument why that is true. If it is false, give an example that violates the statement.

a) All monotonic sequences are convergent.

b) If $\{a_n\}$ converges, then $\{\frac{1}{a_n}\}$ does not converge.

c) If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n$ exists.

d) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges absolutely.

a) False. Counterexample: For $a_n = n$ for all n is monotonic but diverges.

b) False. Counterexample: For $a_n = 1$ for all n, $\lim a_n = \lim \frac{1}{a_n} = 1$ both converge.

c) True. This was proven in the lecture and can be found in the book under the name n-th term test for divergence. The argument is:
If $\sum_{n=1}^{\infty} a_n$ converges, then the limit $\lim_{n\to\infty} s_n$ of its partial sums $s_n = a_1 + a_2 + \cdots + a_n$ exists. Since $\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n-1}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - s_{n-1} = 0$, the limit $\lim a_n$ exists (and is zero).

d) True. For $\sum_{n=1}^{\infty}(-1)^n a_n$ to converge absolutely, $\sum_{n=1}^{\infty} |(-1)^n a_n| = \sum_{n=1}^{\infty} |a_n|$ needs to converge. This follows from directly for the absolute convergence of $\sum_{n=1}^{\infty}$, since by the definition of absolute convergence, $\sum_{n=1}^{\infty}$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

Question 2. (2+6 points)

a) State the integral test.

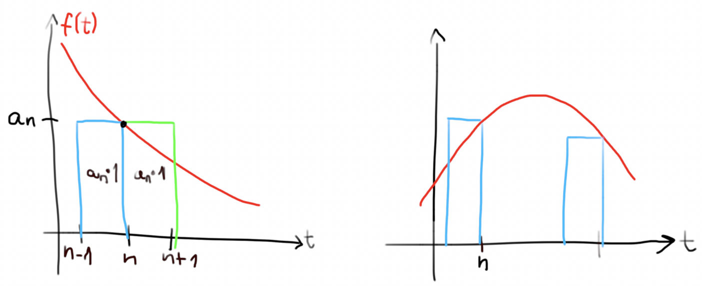
b) The integral test requires the involved function to be continuous and non-increasing. Explain why those conditions are needed. Can they be weakened for the theorem still to hold?

a) Suppose that a function exists that is positive, continuous, and nonincreasing on an interval $[N, \infty)$ for some $0 < N, N \in \mathbb{N}$, and for which function $a_n = f(n)$. Then

$$\sum_{n=1}^{\infty} a_n \qquad \text{and} \qquad \int_N^{\infty} f(t)dt$$

either both converge or both diverge to infinity.

b) Continuity: To ensure that the integral of the function exists. This condition could be weakened to integrability but testing that is more complicated than seeing continuity.
Nonincreasing: The argument that proves the integral test compares the sum given by the sum of area of rectangles with side lengths a_n and 1 to the integral, considered area area under the graph of the function $f(t)$. Since $a_n = f(n)$, and non-increasing function, the rectangle with area $a_n \cdot 1$ lies completely below the graph of the function if its right edge is chosen above n, and the area under the graph on $[n, n+1]$ is contained in the rectangle $a_n \cdot 1$ if its left edge is chosen above n which allows to bound one area by the other. The opposite is true if the function was increasing. (See figure). So if a function is both increasing and decreasing, and the integral is greater than the sum on the increasing parts, then it is smaller than the sum where the function decreases and it is not possible to compare in general.
However, if the function is nondecreasing and negative, the situation is exactly the argument of the proof shows that the integral test also works for negative, continuous, nondecreasing functions. One can also see that by arguing with absolute convergence: If a function exists that is negative, continuous, and nondecreasing on an interval $[N, \infty)$ for some $0 < N, N \in \mathbb{N}$, and for which function $a_n = f(n)$, then its absolute value function $|f(t)|$ is positive, continuous, nonincreasing, and therefore fulfills the conditions for the integral test.It follows that $\sum_{n=1}^{\infty} |a_n|$ converges. That is, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely and therefore converges.



Question 3. (4+4 points)

Determine whether the following series are convergent or divergent. If the series is convergent explain if it is absolutely convergent or conditionally convergent.

$$\text{a)} \quad \sum_{n=1}^{\infty} \frac{\sin(1/n)}{n}, \qquad\qquad \text{b)} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(1+n^2)}.$$

a) Denote $a_n = \frac{\sin(1/n)}{n}$. We use the comparison test with $b_n = \frac{1}{n^2}$. To this purpose we compute

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n \sin(1/n) = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent as a p-series with $p > 1$, the series is convergent as well.
b) Denote $a_n = \frac{(-1)^n}{\ln(1+n^2)}$. For n large enough, we have $|a_n| \geq \frac{1}{n}$ which can be seen as follows: $\lim_{n\to\infty} n|a_n| = \infty$ by L'Hospital $\lim_{n\to\infty} \frac{x}{\ln(1+x^2)} = \lim_{n\to\infty} \frac{1+x^2}{2x} = \lim_{n\to\infty} \frac{1}{2x} + \frac{x}{2} = \infty$. By the comparison test, the series is not absolutely convergent. On the other hand, a_n is alternating and $|a_n|$ is decreasing and converges to 0. By the alternating series test, the series converges. It is therefore conditionally convergent.

Question 4. (4+4 points)

a) Show that the Maclaurin series (Taylor series around 0) of the function $f(x) = \dfrac{3x^2}{1+x^6}$ is given by $\displaystyle\sum_{n=0}^{\infty}(-1)^n 3x^{6n+2}$.

b) Use part (a) to find the Maclaurin series representation of the function $\arctan(x^3)$.

a) Use the Maclaurin series: $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$ for $|t| < 1$.
Substitute $t = -x^6$:

$$\frac{1}{1+x^6} = \sum_{n=0}^{\infty}(-x^6)^n, \text{ for } |x^6| < 1.$$

It follows that

$$\frac{3x^2}{1+x^6} = 3x^2\sum_{n=0}^{\infty}(-1)^n x^6 = \sum_{n=0}^{\infty}(-1)^n 3x^{6n+2}, \text{ for } |x| < 1.$$

b) Note that $\frac{d}{dx}\arctan(x^3) = \frac{3x^2}{1+x^6}$. Then

$$\arctan(x^3) = \int_0^x \sum_{n=0}^{\infty}(-1)^n 3t^{6n+2}dt = \sum_{n=0}^{\infty}(-1)^n \frac{3}{6n+3}t^{6n+3}|_0^x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{6n+1}$$

Question 5. (3+2 points)

Consider the following three vectors

$$\mathbf{u} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 0.5 \\ a \\ 2 \end{pmatrix}.$$

a) For which real number a do the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ span a parallelepiped of volume $V = 2$?

b) Show that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det(\mathbf{u}, \mathbf{v}, \mathbf{w})$.

a) The volume of the parallelepiped spanned by the three vectors is the absolute value of the determinant

$$\begin{bmatrix} 2 & -1 & 0.5 \\ 4 & 2 & a \\ 2 & 3 & 2 \end{bmatrix} = 2(4-3a)-4(-2-1.5)+2(-a-1) = 8-6a+8+6-2a-2 = 20-8a$$

Since $20-8a = 2 \Leftrightarrow 18 = 8a$, The volume of the parallelepiped spanned by the three vectors equals one for $a = \frac{18}{8} = \frac{9}{4}$.

b) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{pmatrix}2\\4\\2\end{pmatrix}\cdot\left(\begin{pmatrix}-1\\2\\3\end{pmatrix}\times\begin{pmatrix}0.5\\a\\2\end{pmatrix}\right) = \begin{pmatrix}2\\4\\2\end{pmatrix}\cdot\begin{pmatrix}4-3a\\3.5\\-a-1\end{pmatrix}$
$= 2(4-3a)+2(-a-1)+14 = 8-6a-2a-2+14 = 20-8a$ gives the same result as the computation in a).

Question 6. (1+2+4+4 points)
Let P_1 and P_2 be planes and p be a point

$$\mathbf{P_1} = \begin{pmatrix}2\\3\\2\end{pmatrix} + \lambda_1\begin{pmatrix}1\\3\\2\end{pmatrix} + \lambda_2\begin{pmatrix}1\\1\\0\end{pmatrix}, \qquad \mathbf{P_2} = \begin{pmatrix}0\\-1\\0\end{pmatrix} + \mu_1\begin{pmatrix}1\\1\\1\end{pmatrix} + \mu_2\begin{pmatrix}0\\-1\\0\end{pmatrix}, \qquad \mathbf{p} = (9,2,-9).$$

a) Give the equation of P_1 in standard form $Ax + By + Cz = D$.

b) Give the equation of P_2 in normal form $(\mathbf{x}-\mathbf{v})\cdot\mathbf{n} = 0$.

c) The planes P_1 and P_2 intersect. Give the equation of the line of intersection.

d) Compute the distance between p and the line of intersection.

a) $-(x-2)+(y-3)-(z-2) = 0 \leftrightarrow -x+y-z = -1$.

b) $\left(\begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}0\\-1\\0\end{pmatrix}\right)\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix} = 0 \Leftrightarrow x - z = 0.$

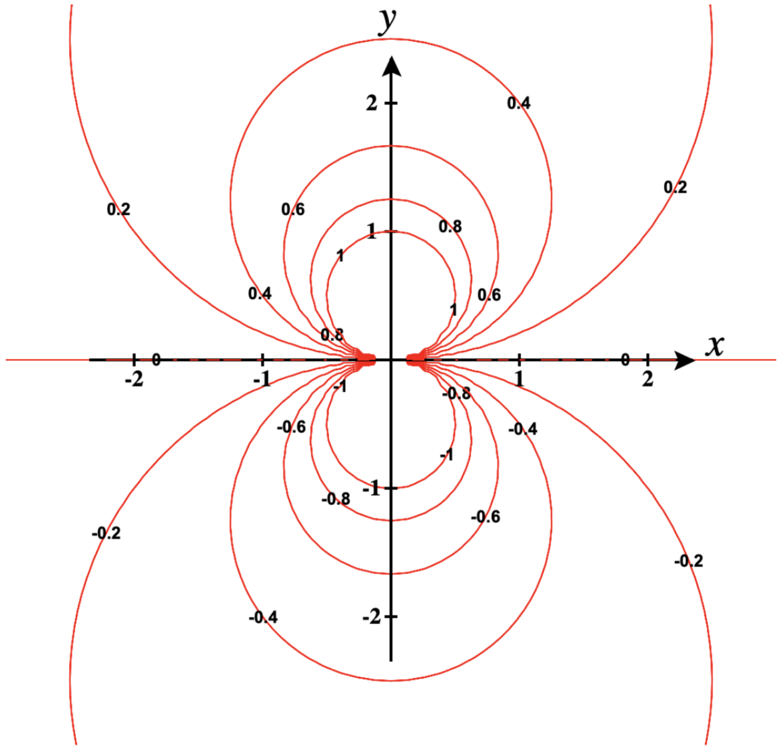
c) Since for points of $P_2 : x = z$ with the equation for P_1 we get $-x+y-x = -1 \Leftrightarrow y = -1+2x$ and the points that are on both planes are given by the line $L = \left\{\begin{pmatrix}0\\-1\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\1\end{pmatrix}, \lambda\in\mathbb{R}\right\}$

d) To obtain the distance, get the line that runs through p and intersects L orthogonally. The lengths of the line segment between the intersection point of the two lines and the point p is the distance.
The point of intersection of the two lines is found by finding λ such that
$\left(\begin{pmatrix}9\\2\\-9\end{pmatrix} - \begin{pmatrix}1\\-1+2\lambda\\1\end{pmatrix}\right)\cdot\begin{pmatrix}1\\2\\1\end{pmatrix} = 0 \Leftrightarrow 8+6-4\lambda-10 = 0 \Leftrightarrow \lambda = 1.$
Therefore, the distance is the length of the vector $\begin{pmatrix}9\\2\\-9\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}8\\1\\-10\end{pmatrix} = \sqrt{64+1+100} = \sqrt{165}.$

Question 7. (1+3 points)

a) Sketch some of the level curves of the function $f(x,y) = \dfrac{y}{x^2+y^2}$.

b) Determine the domain of the function.



a)

b) The domain is $\{(x, y)|(x, y) \neq (0, 0)\}$.

Question 8. (5 points)
Consider the function $f(x) = x^2 - xy$. Show that the partial derivative $\frac{\partial f}{\partial x}$ obtained by using differentiation rules equals the partial derivative computed directly from the definition.

Using differentiation rules: $\dfrac{\partial f}{\partial x} = 2x - y^2$.
Using the definition:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - (x+h)y^2 - (x^2 - xy^2)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - xy^2 - hy^2 - x^2 + xy^2}{h} = \lim_{h \to 0} 2x^2 - y^2 = 2x - y^2$$

End of exam.