| VU Amsterdam | Calculus 2 for BA (X_400636) |
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| Faculty of Sciences | Exam 1 |
| Dr. Senja Barthel | 21-11-2022, 15:30-17:45 (+30 minutes extra time) |

**The use of a calculator, the book, or lecture notes is not permitted.**
**Do not just give answers, but write calculations and explain your steps.**

**You can score 51 points.** $\textbf{Grade} = \dfrac{3 \cdot \textbf{Points}}{17} + 1$

**Question 1.** (2+2+2+2 points)

For the following statements, decide whether they are true or false. If a statement is true, give an argument why that is true. If it is false, give an example that violates the statement.

a) All convergent sequences are monotonic.

b) If neither $a_n$ nor $b_n$ converge, then $a_n b_n$ does not converge.

c) If $a_n = 0$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n$ converges.

d) If $\lim_{n\to\infty} a_n$ exists, then $\sum_{n=1}^{\infty} a_n$ converges.

a) False. Counterexample: The alternating harmonic series is alternating and therefore not monotonic, but convergent nevertheless.

b) False. Choose $a_n$ with $a_n = (-1)^n$ and $b_n$ with $b_n = (-1)^{n+1}$ for all $n \in \mathbb{N}$. Neither of the two sequences converge.
However, $a_n b_n = (-1)^n(-1)^{n+1} = (-1)^{2n+1} = -1$ for all $n$ converges to $-1$.

c) Correct. The sum whose all summands is zero is zero itself and therefore converges: $\sum_{n=1}^{\infty} 0 = 0$.

d) False. For a counterexample choose $a_n$ with $a_n = 1$ for all $n \in \mathbb{N}$.
Then $\lim_{n\to\infty} a_n = 1$ exists but $\sum_{n=1}^{\infty} 1$ diverges to infinity.

**Question 2.** (2+3 points)

The alternating series test states that a series $\sum_{n=1}^{\infty} a_n$ converges if

(i) $a_n a_{n+1} < 0$ for all $n$

(ii) $|a_n| > |a_{n+1}|$ for all $n$

(iii) $\lim_{n\to\infty} a_n = 0$

Answer the following questions about the alternating series test:

a) Is the statement still true if $\{a_n\}$ is not alternating? Justify your answer.

b) Describe a step in the proof of the statement that relies on the second condition $|a_n| > |a_{n+1}|$.

a) No, the statement is not true in general if the first condition is dropped. The harmonic series fulfils (ii) and (iii) but nevertheless diverges.

b) The condition $|a_n| > |a_{n+1}|$ is needed in the proof of the statement to show that the partial sequences $\{s_{2n}\}$ and $\{s_{2n+1}\}$ decrease and increase respectively. Furthermore, the condition $|a_n| > |a_{n+1}|$ allows to find lower respectively upper bounds for the two subsequences.

**Question 3.** (2 points, 4 points, 4 points)

a) Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\frac{1}{3^n} n^2}$ convergent or divergent?

b) Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{17 + \sin(n + 5\pi)}{n^2 + 3n + 1}$ convergent or divergent?

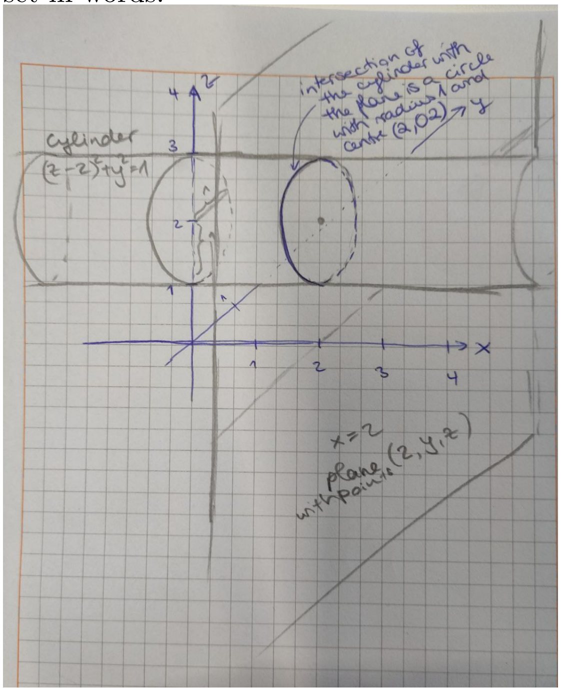
c) For the power series $\displaystyle\sum_{k=1}^{\infty} \frac{2^k (x+1)^k}{\cos\left(\frac{1}{k}\right)}$ determine the radius and centre of convergence.

a) Quotient test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1} n^2}{3^n (n+1)^2} = 3 \frac{n^2}{(n+1^2)} \overset{n\to\infty}{\longrightarrow} 3 > 1$ , therefore the series diverges.

b) $\left|\frac{17+\sin(n+5\pi)}{n^2+3n+1}\right| \leq \frac{18}{n^2}$. Since the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series), the series $\displaystyle\sum_{n=1}^{\infty} \frac{17 + \sin(n + 5\pi)}{n^2 + 3n + 1}$ converges by the comparison criterion.

c) $R = \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty} \left|\frac{2^k \cos\left(\frac{1}{k+1}\right)}{2^{k+1} \cos\left(\frac{1}{k}\right)}\right|$
$= \lim_{k\to\infty} \frac{1}{2} \left|\frac{\cos\left(\frac{1}{k+1}\right)}{\cos\left(\frac{1}{k}\right)}\right| = \frac{1}{2}$. The centre of convergence is $c = -1$.

**Question 4.** (4 points)
Sketch the set of points in $\mathbb{R}^3$ that satisfy $x = 2$ and $(z-2)^2 + y^2 = 1$ and describe the set in words.




Correct cylinder , correct plane. Correct circle drawn. The set of intersection is a circle with radius 1 and centre $(2, 0, 2)$.

**Question 5.** (3+2 points)

Consider the following two vectors

$$\mathbf{u} = \begin{pmatrix} 8 \\ a \\ -2 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} a \\ 2 \\ 10 \end{pmatrix}.$$

a) For which real numbers $a$ are the vectors $\mathbf{u}$ and $\mathbf{v}$ orthogonal with respect to the standard scalar product of $\mathbb{R}^3$?

b) Describe the geometrical meaning of $\mathbf{u} \times \mathbf{v}$.

a) The vectors are orthogonal if their scalar product is zero:

$$\left( \begin{pmatrix} 8 \\ a \\ -2 \end{pmatrix} \cdot \begin{pmatrix} a \\ 2 \\ 10 \end{pmatrix} \right) = 8a + 2a - 20 = 10a - 20 \Rightarrow a = 2.$$

Therefore, the vectors are orthogonal for $a = 2$.

b) The absolute value of the vector product of two vectors describes the area of the parallelogram that the two vectors span. The sign of the vector product describes the ordering of the vectors: $\mathbf{u} \times \mathbf{v}$ is positive if the inner angle of the parallelogram spanned by the two vectors is between $\mathbf{u}, \mathbf{v}$ (starting from $\mathbf{u}$ to reach $\mathbf{v}$ counter-clockwise), and negative otherwise.

**Question 6.** (3 points)

Compute the angle between the vectors $\mathbf{v} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

The scalar product is related to the angle $\alpha$ between two vectors $\mathbf{v}$ and $\mathbf{w}$ by $|\mathbf{v}||\mathbf{w}|\cos(\alpha) = \mathbf{v} \cdot \mathbf{w}$. For the vectors given here, this becomes

$$\cos(\alpha) = \frac{\begin{pmatrix} 3 \\ 9 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} 3 \\ 9 \end{pmatrix}\right| \cdot \left|\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right|} = \frac{15}{\sqrt{3^2+9^2}\sqrt{2^2+1^2}} = \frac{15}{\sqrt{90 \cdot 5}} = \frac{15}{15\sqrt{2}} = \frac{1}{\sqrt{2}}$$

The angle is given by $\alpha = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$.

**Question 7.** (1+2+4 points)

Consider the line $l = p + \lambda \mathbf{v}$, and the points $A$, $B$ with

$$p = (-6, -4, 1), \qquad \mathbf{v} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}, \qquad A = (2, 2, 4), \qquad B = (-1, 6, 3).$$

a) Give an equation for the plane $P$ that contains the line $l$ and the point $A$.

b) Give an equation for the line $L$ that is perpendicular to the plane $P$ and that runs through the point $B$.

c) Compute the distance between the point $B$ and the plane $P$.

a) One directional vector of the plane is $\mathbf{v}$, a second one is given by $A - p$:

$$P = \begin{pmatrix} -6 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 8 \\ 6 \\ 3 \end{pmatrix}, \lambda, \mu \in \mathbb{R}.$$

Since $3 \neq -2 \cdot 2 = -4$, the two directional vectors are not multiples of one another and do span a plane.

b) Vector perpendicular to $P$: $\begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 8 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 21 \\ -28 \\ 0 \end{pmatrix}$

Shorter vector in same direction: $\begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$ (not needed to shorten the vector but makes it easier to handle). Line $L$ is given by $L = \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix} + \kappa \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}, \kappa \in \mathbb{R}$

c) The distance between $B$ and the plane $P$ is the length of the vector with endpoints $B$ and $p$ projected on directional vector of the line $L$ of length one:

$$d = \left| \frac{\left[\begin{pmatrix} -6 \\ -4 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix}\right] \cdot \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}}{\left|\begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}\right|} \right| = \frac{\left|\begin{pmatrix} -5 \\ -10 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}\right|}{\sqrt{3^2 + (-4)^2 + 0^2}} = \frac{-15+40}{\sqrt{25}} = \frac{25}{5} = 5$$

**Question 8.** (1+4 points)

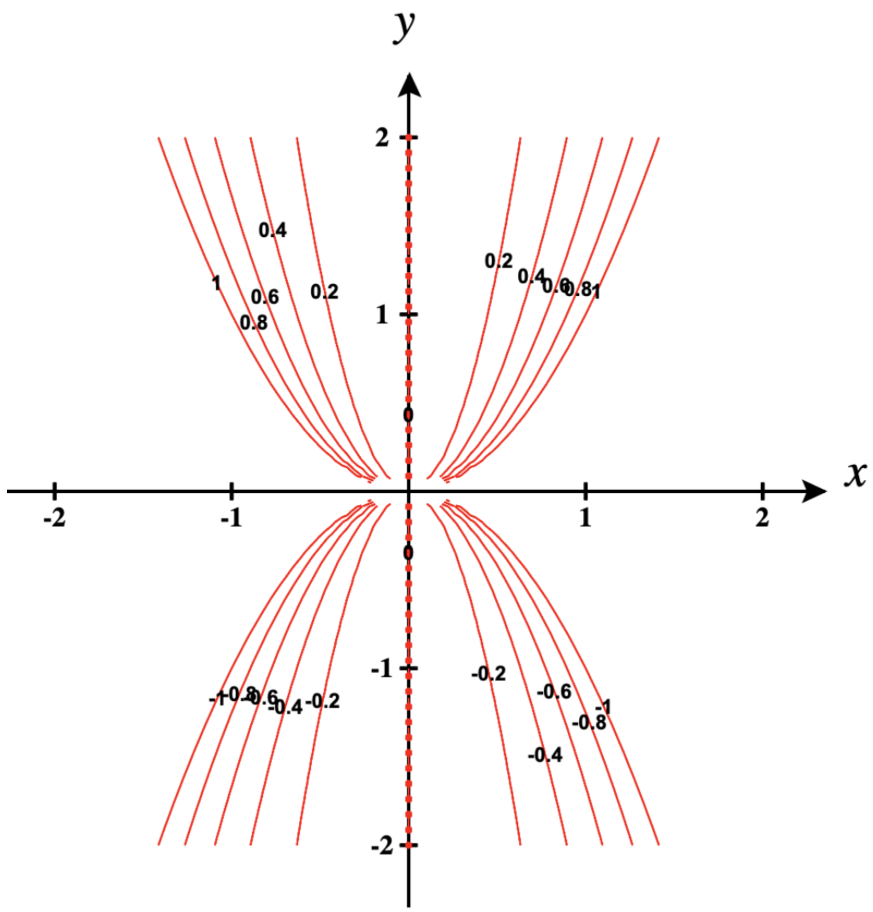
Consider the function $f(x, y) = \dfrac{x^2}{y}$.

a) Determine the domain of the function.

b) Sketch some of the level curves of the function, indicate estimate values of the function taken on the level curves.

a) The domain of the function is $\{(x, y) \in \mathbb{R}^2, y \neq 0\}$

b) Level curves: (1 point for parabolic shapes, 1 point for them to be correctly stretched, (1 point subtracted if the lines intersect the x-axis anywhere - including at the origin since the function is not defined for y=0) 1 point for absolute values increasing to more opened curves, 1 point for indicating negative and positive values,)




**Question 9.** (2+2 points)

The continuous function $f(x, y) : R^2 \to R$ is defined as

$$f(x, y) = \begin{cases} \frac{y^3}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}.$$

Compute the partial derivatives for all points $(x, y)$ in which the partial derivatives exist. (You can use differentiation rules as well as the definition for the computations.)

The partial derivatives exist for $(x,y) \neq (0,0)$ since the function is a concatenation of partial differentiable functions:

$$\frac{\partial f}{\partial x}(x,y) = -\frac{2xy^3}{(x^2+y^2)^2}, \; \frac{\partial f}{\partial y}(x,y) = \frac{3y^2x^2+y^4}{(x^2+y^2)^2}$$

For $(x,y) = (0,0)$:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = 0; \; \frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{t^3}{t^2}}{t} = 1$$

**End of exam.**