VU Amsterdam	Calculus 2 for BA (X_400636)
Faculty of Sciences	Second Test
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The use of a calculator, the book, or lecture notes is <u>not</u> permitted.

Do not just give answers, but write calculations and explain your steps.

You can score 36 points. Grade=(Points/4)+1

Question 1. (4 points, 2 points)

Consider the function

$$f(x,y) = \frac{x}{y} + \ln(1+xy^2)$$

a) Find the rate of change of f at the point $(2,1)$ in the direction of the vector $\begin{pmatrix}1\\1\end{pmatrix}$.
Find the maximum rate of increase of f at the point $(2,1)$.

b) Compute $\dfrac{\partial^2 f}{\partial x \partial y}$ at the point $(2,1)$.

Solution.

a) We compute the two partial derivatives of f at an arbitrary point

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{y} + \frac{y^2}{1+xy^2}, \quad \textbf{(0.5P)} \qquad \frac{\partial f}{\partial y}(x,y) = -\frac{x}{y^2} + \frac{2xy}{1+xy^2}. \quad \textbf{(0.5P)}$$

Evaluating at $x=2$ and $y=1$, we find

$$\frac{\partial f}{\partial x}(2,1) = \frac{4}{3}, \quad \textbf{(0.5P)} \qquad \frac{\partial f}{\partial y}(2,1) = -\frac{2}{3}. \quad \textbf{(0.5P)}$$

The rate of change f at $(2,1)$ in the direction of $\mathbf{v} := \begin{pmatrix}1\\1\end{pmatrix}$ is

$$D_{\mathbf{v}/|\mathbf{v}|} f(2,1) = \frac{\mathbf{v}}{|\mathbf{v}|} \bullet \nabla f(2,1) = \frac{1}{\sqrt{2}}\Big(1\cdot\frac{\partial f}{\partial x}(2,1) + 1\cdot\frac{\partial f}{\partial y}(2,1)\Big) = \frac{\sqrt{2}}{3}. \qquad \textbf{(1P)}$$

The maximum rate of increase of f at the point $(2,1)$ is

$$|\nabla f(2,1)| = \sqrt{\Big(\frac{\partial f}{\partial x}(2,1)\Big)^2 + \Big(\frac{\partial f}{\partial y}(2,1)\Big)^2} = \frac{2\sqrt{5}}{3}. \qquad \textbf{(1P)}$$

b) We compute

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial y}\Big) = -\frac{1}{y^2} + \frac{2y}{(1+xy^2)^2} \qquad \textbf{(1P)}$$

and evaluating at $(2,1)$:

$$\frac{\partial^2 f}{\partial x \partial y}(2,1) = -\frac{7}{9}. \qquad \textbf{(1P)}$$

Question 2. (4 points, 3 points)

The function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f(x,y) = -xy^2 + y^3 + y^2 + \frac{x^2}{2}.$$

a) Determine all critical points of f.

b) Classify the two critical points $(0,0)$ and $(4,2)$.

Solution.
a) The critical points (x,y) of f satisfy the system of equations

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = 0 \\ \dfrac{\partial f}{\partial y}(x,y) = 0. \end{cases} \qquad \textbf{(0.5P)}$$

We compute the partial derivatives of f:

$$\frac{\partial f}{\partial x}(x,y) = -y^2 + x, \quad \textbf{(0.5P)} \qquad \frac{\partial f}{\partial y}(x,y) = -2xy + 3y^2 + 2y. \quad \textbf{(0.5P)}$$

Therefore the system of equations is

$$\begin{cases} -y^2 + x = 0 \\ -2xy + 3y^2 + 2y = 0. \end{cases}$$

From the first one we get $x = y^2$, which substituted into the second one yields

$$y(-2y^2 + 3y + 2) = 0, \qquad \textbf{(0.5P)}$$

which implies $y = 0$ **(0.5P)** or $2y^2 - 3y - 2 = 0$.
The equation $2y^2 - 3y - 2 = 0$ has the solutions $y = 2$ and $y = -1/2$ **(0.5P)**. Since $x = y^2$, the critical points are

$$(0,0), \quad (4,2), \quad (\tfrac{1}{4}, -\tfrac{1}{2}). \qquad \textbf{(1P)}$$

b) We compute the second partial derivatives of f

$$A = \frac{\partial^2 f}{\partial x^2} = 1, \qquad B = \frac{\partial^2 f}{\partial x \partial y} = -2y, \qquad C = \frac{\partial^2 f}{\partial y^2} = -2x + 6y + 2. \qquad \textbf{(1P)}$$

For $(x,y) = (0,0)$, we get $A > 0$ and $B^2 - AC = -2 < 0$. Therefore, $(0,0)$ is a local minimum **(1P)**. For $(x,y) = (4,2)$, we get $A > 0$ and $B^2 - AC = 10 > 0$. Therefore, $(4,2)$ is a saddle point **(1P)**.

Question 3. (4 points)

Use the method of Lagrange multipliers to find the minimum and maximum value of the function $f(x,y) = xy - y$ subject to the constraint $x^2 + y^2 = 1$.

Solution. The minimum and maximum value are attained at points (x,y) such that there is a λ satisfying the system of equations

$$\begin{cases} 0 = \dfrac{\partial L}{\partial x}(x,y,\lambda) \\ 0 = \dfrac{\partial L}{\partial y}(x,y,\lambda) \\ 0 = \dfrac{\partial L}{\partial \lambda}(x,y,\lambda). \end{cases} \qquad \textbf{(0.5P)}$$

where $L(x, y, \lambda) = L(x, y, \lambda) = xy - y + \lambda(x^2 + y^2 - 1)$ **(0.5P)**. We compute

$$\frac{\partial L}{\partial x}(x, y, \lambda) = y + 2\lambda x, \quad \frac{\partial L}{\partial y}(x, y, \lambda) = x - 1 + 2\lambda y, \quad \frac{\partial L}{\partial \lambda}(x, y, \lambda) = x^2 + y^2 - 1. \quad \textbf{(1P)}$$

Thus we get the system

$$\begin{cases} 0 = y + 2\lambda x \\ 0 = x - 1 + 2\lambda y \\ 0 = x^2 + y^2 - 1. \end{cases}$$

Eliminating λ from the first two equations, we get

$$y^2 = x^2 - x, \qquad \textbf{(0.5P)}$$

which substituted in the last equation yields

$$2x^2 - x - 1 = 0, \qquad \textbf{(0.5P)}$$

which has $x = 1$ and $x = -\frac{1}{2}$ as solutions. The first one yields $y^2 = 0$ and the second one $y^2 = \frac{3}{4}$. Therefore, the solutions are

$$(1, 0), \qquad (-\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}), \qquad (-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}). \qquad \textbf{(0.5P)}$$

There holds

$$f(1, 0) = 0, \qquad f(-\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}) = \tfrac{3\sqrt{3}}{4}, \qquad f(-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}) = -\tfrac{3\sqrt{3}}{4}.$$

Therefore, the minimum value is $-\frac{3\sqrt{3}}{4}$ and the maximum value is $\frac{3\sqrt{3}}{4}$ **(0.5P)**.

Question 4. (3 points)

Compute

$$\int_0^{\sqrt[3]{\pi^2}} \int_{\sqrt{y}}^{\sqrt[3]{\pi}} 3\sin(x^3)\, dx\, dy.$$

Solution. We have

$$I = \int_0^{\sqrt[3]{\pi^2}} \int_{\sqrt{y}}^{\sqrt[3]{\pi}} 3\sin(x^3)\, dx\, dy = \iint_D 3\sin(x^3) dA,$$

where D is the set of points (x, y) such that $0 \leq y \leq \sqrt[3]{\pi^2}$ and $\sqrt{y} \leq x \leq \sqrt[3]{\pi}$. This means that $x \in [0, \sqrt[3]{\pi}]$ and that $0 \leq y \leq x^2$. Therefore, we have

$$I \overset{\textbf{(1P)}}{=} \int_0^{\sqrt[3]{\pi}} \int_0^{x^2} 3\sin(x^3)\, dy\, dx \overset{\textbf{(1P)}}{=} \int_0^{\sqrt[3]{\pi}} 3\sin(x^3)x^2\, dx = -\cos(x^3)\Big|_0^{\sqrt[3]{\pi}} \overset{\textbf{(1P)}}{=} 2.$$

Question 5. (4 points)

Let R be the finite region in the first quadrant of the xy-plane bounded by the line $y = 0$, the line $\sqrt{3}y = x$ and the curve $x^2 + y^2 = 4$. Compute

$$\iint_R 5xy^2\, dA.$$

Solution. Let (r, θ) denote polar coordinates. The points in the region R satisfy

$$0 \le r \le 2, \qquad 0 \le \theta \le \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} \qquad \textbf{(1P)}.$$

Therefore, we get

$$I = \iint_R 5xy^2 \, dA = \int_0^{\frac{\pi}{6}} \int_0^2 5(r\cos\theta)(r\sin\theta)^2 r \, dr d\theta = \int_0^{\frac{\pi}{6}} \int_0^2 5r^4 \cos\theta \sin^2\theta \, dr d\theta. \quad \textbf{(1P)}$$

We integrate with respect to r

$$I = \int_0^{\frac{\pi}{6}} r^5 \cos\theta \sin^2\theta \Big|_{r=0}^{r=2} d\theta = 32 \int_0^{\frac{\pi}{6}} \cos\theta \sin^2\theta \, d\theta \qquad \textbf{(1P)}$$

and with respect to θ

$$I = 32 \cdot \frac{1}{3} \sin^3\theta \Big|_0^{\frac{\pi}{6}} = \frac{4}{3}. \qquad \textbf{(1P)}$$

Question 6. (3 points)

Transform the polar equation

$$r = \frac{1}{\sqrt{1 + 2\cos(2\theta)}}$$

to rectangular coordinates, and describe the curve represented.

Solution. We square both sides of the polar equation (since they are positive) and get rid of denominators obtaining

$$1^2 \overset{\textbf{(0.5P)}}{=} r^2(1 + 2\cos(2\theta)) \overset{\textbf{(0.5P)}}{=} r^2 + 2r^2(\cos^2\theta - \sin^2\theta),$$

where we used the duplication formula for the cosine. Rectangular coordinates (x, y) and polar coordinates are related by $x = r\cos\theta$ and $y = r\sin\theta$. In particular, $x^2 + y^2 = r^2$. Therefore,

$$r^2 + 2r^2(\cos^2\theta - \sin^2\theta) = (x^2 + y^2) + 2x^2 - 2y^2 = 3x^2 - y^2 \qquad \textbf{(0.5P)}$$

and the equation in rectangular coordinates is

$$3x^2 - y^2 = 1. \qquad \textbf{(0.5P)}$$

Thus, the curve represented is a hyperbola **(1P)** (with asymptotes $\sqrt{3}x = \pm y$).

Question 7. (2 points, 1 point)

a) Write the polar representation of all complex numbers z satisfying $z^3 = 2 + 2i$.

b) Compute the real and imaginary part of all complex numbers z satisfying $z^3 = 2 + 2i$ and belonging to the second quadrant of the complex plane.

Solution.
a) We have

$$|2+2i| = \sqrt{2^3}, \qquad \arg(2+2i) = \frac{\pi}{4} + 2\pi k,\ k \in \mathbb{Z}. \qquad \textbf{(1P)}$$

Therefore, the solutions to $z^3 = 2+2i$ have

$$|z| = (\sqrt{2^3})^{\frac{1}{3}} = \sqrt{2}, \qquad \arg(z) = \frac{\pi}{12} + \frac{2\pi}{3}k,\ k \in \mathbb{Z}. \qquad \textbf{(0.5P)}$$

For $k=0$, we get $\arg(z) = \frac{\pi}{12} \in [0, 2\pi)$. For $k=1$, we get $\arg(z) = \frac{3\pi}{4} \in [0, 2\pi)$. For $k=2$, we get $\arg(z) = \frac{17\pi}{12} \in [0, 2\pi)$. Therefore, we obtain three solutions

$$\textbf{(0.5P)} \quad \begin{cases} z_0 = \sqrt{2}\Big(\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\Big), \\ z_1 = \sqrt{2}\Big(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\Big), \\ z_2 = \sqrt{2}\Big(\cos\left(\frac{17\pi}{12}\right) + i\sin\left(\frac{17\pi}{12}\right)\Big) \end{cases}$$

b) If $\arg(z) \in [0, 2\pi)$, then z belongs to the second quadrant exactly if $\frac{\pi}{2} \leq \arg(z) \leq \pi$ **(0.5P)**. Therefore, only z_1 belongs to the second quadrant and we can write

$$z_1 = \sqrt{2}\Big(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\Big) = -1 + i. \qquad \textbf{(0.5P)}$$

Question 8. (3 points)

Find the function $y(x)$ solving the initial-value problem

$$\begin{cases} \dfrac{dy}{dx} = 2xe^{x^2 - y}, \\ y(0) = \ln 2. \end{cases}$$

Solution. Step 1: We separate the variables in the equation and get

$$e^y \frac{dy}{dx} = 2xe^{x^2}. \qquad \textbf{(0.5P)}$$

Step 2: Using the chain rule, we see that this equation is equivalent to

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(e^{x^2}). \qquad \textbf{(1P)}$$

Step 3: Integrating we get $e^y = e^{x^2} + C$ for some $C \in \mathbb{R}$ and therefore

$$y(x) = \ln(e^{x^2} + C), \quad C \in \mathbb{R}. \qquad \textbf{(1P)}$$

Step 4: Substituting $x = 0$, we find $\ln 2 = \ln(1+C)$. Hence $C = 1$ and the solution is

$$y(x) = \ln(e^{x^2} + 1). \quad \textbf{(0.5P)}$$

For Step 2 and 3 there is also the following alternative solution:

$$\int e^y \frac{dy}{dx}\, dx = \int 2xe^{x^2}\, dx \qquad \Longleftrightarrow \qquad \textbf{(0.5P)}$$

$$\int e^y\, dy = \int e^u\, du, \qquad u = x^2 \qquad \Longleftrightarrow \qquad \textbf{(0.5P)}$$

$$e^y = e^{x^2} + C \qquad \Longleftrightarrow \qquad \textbf{(0.5P)}$$

$$y = \ln(e^{x^2} + C) \qquad \textbf{(0.5P)}$$

Question 9. (3 points)

Find the function $y(x)$ solving the initial-value problem

$$\begin{cases} y'' - 2y' + 5y = 0, \\ y(0) = 2, \\ y'(0) = 0. \end{cases}$$

Solution. We solve the associated quadratic equation $k^2 - 2k - 5 = 0$ **(0.5P)**. Its solutions are $k = 1 \pm 2i$ **(0.5P)**. Therefore, the general solution of $y'' - 2y' + 5y = 0$ is

$$y(x) = Ae^x \cos(2x) + Be^x \sin(2x), \; A, B \in \mathbb{R}. \qquad \textbf{(1P)}$$

We have $y(0) = A$ and $y'(0) = A + 2B$. Therefore, the initial conditions imply $A = 2$ **(0.5P)** and $B = -1$ **(0.5P)**. The desired solution is

$$y(x) = 2e^x \cos(2x) - e^x \sin(2x).$$