Resit Calculus 2, 6 February 2020, Solutions

Guideline for corrections:

- minor mistake (for example a computational error): substract $\frac{1}{2}$ point;
- major mistake (for example a conceptual error): substract 1 point;
- answer written somewhere but not clearly articulated: subtract $\frac{1}{2}$ point;
- correct answer but derivation/motivation not clear: subtract 1 point.

1. a) The sequence $\frac{(-1)^n\sqrt{n}}{\ln n}$ does not converge to zero, so the series is divergent. [**1 point**]

   b) Write $a_n = \frac{(-1)^{n-1}}{n^3}$. Since $\sum_{n=1}^{\infty} |a_n|$ is a convergent p-series, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. [**1 point**]

2. We write the power series as $\sum_{n=0}^{\infty} a_n(x-3)^n$ where $a_n = \frac{2^n}{\sqrt{n+3}}$. Then

$$L := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}}{\sqrt{n+4}} \frac{\sqrt{n+3}}{2^n}\right| = 2.$$

It follows that the radius of convergence is $R = 1/L = \frac{1}{2}$. [**1 point**] We now investigate the boundary points, $x = 3 - 1/2 = 5/2$ and $x = 3 + 1/2 = 7/2$. For $x = 5/2$, the series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$ which is conditionally convergent by the Alternating Series Theorem. [**1 point**] For $x = 7/2$, the series is $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+3}}$ which is a divergent p-series (with $p = 1/2$). [**1 point**] We find that the interval convergence is $[5/2, 7/2)$. [**-1/2 point if the final conclusion is not stated**]

3. Using the standard series $\exp(y) = \sum_{n=0}^{\infty} \frac{y^n}{n!}$, we find

$$10^x = 10 \times 10^{x-1} = 10\exp(\ln(10)(x-1)) = 10\sum_{n=0}^{\infty} \frac{(\ln 10)^n (x-1)^n}{n!}. \qquad [\text{ 1 point }]$$

(An alternative way is to differentiate $f(x) = 10^x = \exp(\ln(10)x)$, resulting in

$$f'(x) = \ln(10)\exp(\ln(10)x) = \ln(10)f(x),$$

and by repeating, $f^{(n)}(x) = (\ln(10))^n f(x)$. It follows that

$$f^{(n)}(1) = 10(\ln(10))^n.$$

Using the general formula for the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

gives the same result as above.)

The radius of convergence for the Taylor series of the exponential series $\exp(y)$ is known to be ∞. Setting $y = \ln(10)(x-1)$ gives the radius of convergence $R = \infty$ in terms of x. [**1 point**]

(Alternatively, compute

$$L = \lim_{n\to\infty} \left|\frac{(\ln 10)^{n+1}}{(n+1)!} \frac{n!}{(\ln 10)^n}\right| = 0,$$

resulting in $R = 1/L = \infty$.)

4.

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix} = \mathbf{j} - 5\mathbf{k} \quad \text{and} \quad P = (3, -2, 1).$$

a) $\mathbf{u} \bullet \mathbf{v} = 1 \times 0 + 1 \times 1 + (-2) \times (-5) = 11$ [**1 point**] and

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 1 \times (-5) - (-2) \times 1 \\ (0) \times (-2) - (1) \times (-5) \\ (1) \times (1) - (1) \times (0) \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}. \quad [\ \textbf{1 point}\ ]$$

b)

$$\mathbf{u_v} = \left(\frac{\mathbf{u} \bullet \mathbf{v}}{\mathbf{v} \bullet \mathbf{v}}\right) \mathbf{v} = \frac{11}{0^2 + 1^2 + (-5)^2} \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix} = \frac{11}{26} \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix} = \frac{11}{26}\mathbf{j} - \frac{55}{26}\mathbf{k}.$$

c) This equation is given by

$$(x - 3) + (y + 2) - 2(z - 1) = 0, \quad [\ \textbf{1 point}\ ],$$

which can also be written as

$$z = 1 + \tfrac{1}{2}(x - 3) + \tfrac{1}{2}(y + 2).$$

5. We compute

$$\frac{\partial}{\partial y} f(xy^2, xy) = 2xy f_1(xy^2, xy) + x f_2(xy^2, xy), \quad [\ \textbf{1 point}\ ],$$

and therefore

$$\begin{aligned} \frac{\partial^2}{\partial y^2} f(xy^2, xy) &= \frac{\partial}{\partial y} \left(2xy f_1(xy^2, xy) + x f_2(xy^2, xy)\right) \\ &= 2x f_1(xy^2, xy) + (2xy)^2 f_{11}(xy^2, xy) + 2x^2 y f_{12}(xy^2, xy) \\ &\quad + 2x^2 y f_{21}(xy^2, xy) + x^2 f_{22}(xy^2, xy), \quad [\ \textbf{1 point}\ ] \end{aligned}$$

which (since the partial derivatives f_{12} and f_{21} are equal under the stated conditions) can also be written e.g. as

$$\frac{\partial^2}{\partial y^2} f(xy^2, xy) = 2x f_1(xy^2, xy) + (2xy)^2 f_{11}(xy^2, xy) + 4x^2 y f_{12}(xy^2, xy) + x^2 f_{22}(xy^2, xy).$$

6.

$$f(x, y) = -x^2 - xy^2 + y^2 + 6x - 2.$$

a) We compute the gradient of f,

$$\nabla f(x, y) = \begin{pmatrix} -2x - y^2 + 6 \\ -2xy + 2y \end{pmatrix}. \quad [\ \textbf{1 point}\ ]$$

Setting the gradient to zero gives after some algebra the critical points

$$(3, 0), \quad (1, -2), \quad (1, 2). \quad [\ \textbf{1 point}\ ].$$

b) The Hessian matrix is given by

$$\nabla^2 f(x,y) = \begin{pmatrix} -2 & -2y \\ -2y & -2x+2 \end{pmatrix}.$$

At $(x,y) = (3,0)$ we have $f_{xx}f_{yy} - (f_{xy})^2 = (-2)(-4) - (0)(0) = 8$. Furthermore $f_{xx} = -2$, so that $(3,0)$ is a local maximum. [**1 point**]
At $(x,y) = (1,\pm 2)$ we have $f_{xx}f_{yy} - (f_{xy})^2 = (-2)(0) - (4)^2 < 0$. This means that $(1,-2)$ and $(1,2)$ are saddle points. [**1 point**]
Similar derivativations give for the 'fake' critical points that $(4,1)$ is a local maximum and $(2,2)$ is a saddle point. [**-1 point** if the student works with these when not necessary.]

c) The equation for the tangent plane at $(-1,1)$ is given by

$$z = f(-1,+1) + \partial_x f(-1,1)(x-(-1)) + \partial_y f(-1,1)(y-1),$$

which gives

$$z = -7 + 7(x+1) + 4(y-1). \quad [\ \textbf{1 point}\ ]$$

7. a) By changing the order of integration,

$$\int_0^1 \left(\int_{y\sqrt{\pi}}^{\sqrt{\pi}} \sin(x^2)\,dx \right) dy = \int_0^{\sqrt{\pi}} \left(\int_0^{x/\sqrt{\pi}} \sin(x^2)\,dy \right) dx \quad [\ \textbf{1 point}\ ]$$
$$= \int_0^{\sqrt{\pi}} (x/\sqrt{\pi}) \sin(x^2)\,dx$$
$$= -\frac{1}{2\sqrt{\pi}} \cos(x^2)\Big|_{x=0}^{x=\sqrt{\pi}} = \frac{1}{\sqrt{\pi}}. \quad [\ \textbf{1 point}\ ]$$

b) By using polar coordinates

$$\int\int_S \frac{x}{\sqrt{x^2+y^2}}\,dA = \int_{-\pi/2}^{\pi/2} \int_0^{\sqrt{2}} \frac{r\cos\theta}{r}\, r\,dr\,d\theta \qquad [\ \textbf{1 point}\ ]$$
$$= \left(\tfrac{1}{2}r^2\right)\Big|_{r=0}^{r=\sqrt{2}} (\sin\theta)\Big|_{\theta=-\pi/2}^{\theta=\pi/2} = 2. \quad [\ \textbf{1 point}\ ]$$

8. We have $\arg(-16\mathrm{i}) = -\pi/2 + k2\pi$ for $k \in \mathbb{Z}$, and $|-16\mathrm{i}| = 16$. [2 × **1/2 point**] Therefore if $z^4 = -16\mathrm{i}$, then $|z| = \sqrt[4]{16} = 2$ and $\arg z = -\pi/8 + k\pi/2$, giving

$$z = 2(\cos(-\pi/8) + \mathrm{i}\sin(-\pi/8)), \qquad z = 2(\cos(3\pi/8) + \mathrm{i}\sin(3\pi/8)),$$
$$z = 2(\cos(7\pi/8) + \mathrm{i}\sin(7\pi/8)) \qquad z = 2(\cos(-5\pi/8) + \mathrm{i}\sin(-5\pi/8). \quad [\ \textbf{1 point}\ ]$$

9. First solve the homogeneous equation $\frac{dy_h}{dx} = xy_h$. Using separation of variables,

$$\frac{1}{y_h}\,dy = x\,dx, \quad \text{giving} \quad \ln|y_h| = x^2/2 + c, \quad \text{i.e.} \quad y_h(x) = Ce^{x^2/2} \quad [\ \textbf{1 point}\ ]$$

for some $C \in \mathbb{R}$. We can now solve the inhomogeneous equation, e.g. using variation of constants, by letting $y(x) = C(x)y_h(x)$. Inserting into the differential equation gives

$$C'(x)e^{x^2/2} + C(x)xe^{x^2/2} - e^{x^2/2}\sin(x) = C(x)xe^{x^2/2},$$

which simplifies to

$$C'(x) = \sin(x), \quad \text{which gives} \quad C(x) = -\cos(x) + C_1$$

for some $C_1 \in \mathbb{R}$. This gives the general solution

$$y(x) = (C_1 - \cos(x))e^{x^2/2}. \quad [\ \textbf{1 point}\ ]$$

Plugging in the initial condition $y(0) = 2$ gives $C_1 = 3$, so that the final solution is given by

$$y(x) = (3 - \cos(x))e^{x^2/2}. \quad [\ \textbf{1 point}\ ]$$