Second test Calculus 2, 18 December 2019, Solutions
Guideline for corrections:

- minor mistake (for example a computational error): substract $\frac{1}{2}$ point;
- major mistake (for example a conceptual error): substract 1 point;
- answer written somewhere but not clearly articulated: subtract $\frac{1}{2}$ point;
- correct answer but derivation/motivation not clear: subtract 1 point.

1.

$$\frac{\partial}{\partial x} f(xy^3, xy) = y^3 f_1(xy^3, xy) + y f_2(xy^3, xy); \qquad \textbf{[1 point]}$$

$$\begin{aligned}\frac{\partial^2}{\partial y \partial x} f(xy^3, xy) &= \frac{\partial}{\partial y}\left(y^3 f_1(xy^3, xy) + y f_2(xy^3, xy)\right)\\ &= 3y^2 f_1(xy^3, xy) + 3xy^5 f_{11}(xy^3, xy) + xy^3 f_{12}(xy^3, xy)\\ &\quad + f_2(xy^3, xy) + 3xy^3 f_{21}(xy^3, xy) + xy f_{22}(xy^3, xy). \qquad \textbf{[2 points]}\end{aligned}$$

Since f has continuous partial derivatives of all orders, the mixed partial derivatives f_{12} and f_{21} are identical and the expression for $\frac{\partial^2}{\partial y \partial x} f(xy^3, xy)$ may be shortened to, for example (but this is not necessary for full points)

$$\begin{aligned}\frac{\partial^2}{\partial y \partial x} f(xy^3, xy) &= 3y^2 f_1(xy^3, xy) + 3xy^5 f_{11}(xy^3, xy) + 4xy^3 f_{12}(xy^3, xy)\\ &\quad + f_2(xy^3, xy) + xy f_{22}(xy^3, xy).\end{aligned}$$

2. $f(x, y) = x^3 + y^3 - 3xy + 1$.

(a) $\nabla f(x, y) = \begin{pmatrix} 3x^2 - 3y \\ 3y^2 - 3x \end{pmatrix}$, and $\widehat{\mathbf{u}} = \frac{1}{\sqrt{3^2+4^2}} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$ so that

$$D_{\widehat{\mathbf{u}}} f(1, 2) = \begin{pmatrix} -3 \\ 9 \end{pmatrix} \bullet \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} = -\frac{9}{5} + \frac{36}{5} = \frac{27}{5}.$$

[$\frac{1}{2}$ **point**] correct gradient/partial derivatives expression in terms of x and y;
[$\frac{1}{2}$ **point**] normalization of $\mathbf{u}$;
[$\frac{1}{2}$ **point**] general formula for directional derivative;
[$\frac{1}{2}$ **point**] correct computation.

(b) Setting $\nabla f(x, y) = 0$ gives the conditions $3x^2 = 3y$ and $3y^2 = 3x$ [**1 point**]. The first equation gives $y = x^2$, and inserting into the second equation gives $x^4 = x$, so that $x = 0$ or $x^3 = 1$. This gives the critical points $(0, 0)$ and $(1, 1)$ [**2 points**].

(c) The Hessian matrix is

$$\nabla^2 f(x, y) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix} \qquad \textbf{[1 point]}.$$

Using the notation $\nabla^2 f(x, y) = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$, we have

$$\det \nabla^2 f(0, 0) = AC - B^2 = 0 \cdot 0 - (-3) \cdot (-3) = -9 < 0,$$

so that it $(0,0)$ is a saddle point **[1 point]**. Next

$$\det \nabla^2 f(1,1) = AC - B^2 = 6 \cdot 6 - (-3) \cdot (-3) = 27 > 0,$$

and

$$\operatorname{tr} \nabla^2 f(1,1) = A + C = 6 + 6 = 12 > 0,$$

so that $(1,1)$ is a local minimum **[1 point]**.
(Similar computations would give for the hypothetical critical points $(\frac{1}{2}, 2)$ and $(0,3)$ that $(\frac{1}{2}, 2)$ is a local minimum and $(0,3)$ is a saddle point.)

3. $f(x,y) = x^2 + y^2$, constraint $xy = 1$. Lagrangian function

$$\mathcal{L}(x,y,\lambda) = x^2 + y^2 + \lambda(xy - 1) \qquad [\tfrac{1}{2}\ \textbf{point}].$$

(It is of course fine if the student works with

$$\mathcal{L}(x,y,\lambda) = x^2 + y^2 + \lambda(1 - xy),$$

the resulting computations should give the same final result.) Computing $\nabla \mathcal{L}(x,y,\lambda)$ and setting to zero gives three necessary conditions:

$$2x + \lambda y = 0, \quad 2y + \lambda x = 0, \quad xy = 1. \qquad [\tfrac{1}{2}\ \textbf{point}]$$

[The third condition is the original constraint, it is not necessary for the student to write it again.] Taking the first equation gives $x = -\frac{1}{2}\lambda y$. Plugging into the second equation gives

$$2y - \tfrac{1}{2}\lambda^2 y = 0,$$

so that either $y = 0$ or $\lambda^2 = 4$. The choice $y = 0$ gives $x = -\frac{1}{2}\lambda y = 0$, which does not satisfy the constraint. The choice $\lambda^2 = 4$ gives $\lambda = \pm 2$, and therefore $x = -y$ or $x = y$. Plugging these into the constraint gives $-y^2 = 1$ (which does not have solutions in $\mathbb{R}$) or $y^2 = 1$, giving $y = \pm 1$. We end up with the critical points $(1,1)$ and $(-1,-1)$. **[1 point]**

(An alternative correct solution deduces $y = 1/x$ from the constraint, and plugs this into the function to obtain $g(x) = f(x^2, 1/x^2) = x^2 + 1/x^2$. The critical points of this function are $x = \pm 1$, leading to the same final result. This is an alternative way to earn **[2 points]**)

The critical points $(-1,-1)$ and $(1,1)$ are necessarily minima: the function $f(x,y)$ constrained to $xy = 1$ does not have a maximum (take $y = 1/x$, then $f(x, 1/x) \to \infty$ as $x \to \infty$). **[1 point]**

4. (a) $y \in [0,1]$ and $x \in [2y, 2]$ is the same as $x \in [0,2]$ and $y \in [0, x/2]$. Therefore

$$\int_0^1 \int_{2y}^2 e^{x^2}\, dx\, dy = \int_0^2 \int_0^{x/2} e^{x^2}\, dy\, dx \qquad \textbf{[1 point]}$$

$$= \int_0^2 \frac{x}{2} e^{x^2}\, dx = \frac{1}{4} e^{x^2} \Big|_{x=0}^{x=2} = \frac{1}{4}\left(e^4 - 1\right). \qquad \textbf{[1 point]}$$

(b) In polar coordinates $S = \{(r,\theta) : 1 \le r \le 3 \,\text{and}\, 0 \le \theta \le \pi/2\}$. Therefore

$$\int\int_S \frac{y}{\sqrt{x^2 + y^2}}\, dA$$

$$= \int_0^{\pi/2} \int_1^3 \frac{r \sin\theta}{r}\, r d\theta, \qquad \textbf{[1 point]}$$

$$= \left(\tfrac{1}{2} r^2 \Big|_{r=1}^{r=3} \right) \cdot \left(-\cos\theta \Big|_{\theta=0}^{\theta=\pi/2} \right) = \left(\frac{9}{2} - \frac{1}{2} \right) \cdot (-0 - (-1)) = 4. \qquad \textbf{[1 point]}$$

5. $z = 1 - \sqrt{3}i$ and $w = 3 + 3i$.

(a)

$$\begin{aligned} |z| &= \sqrt{1^2 + (-\sqrt{3})^2} = 2 && [\tfrac{1}{2}\ \textbf{point}], \\ \operatorname{Arg} z &= \arctan\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3} && [\tfrac{1}{2}\ \textbf{point}], \\ |w| &= \sqrt{3^2 + 3^2} = 3\sqrt{2} && [\tfrac{1}{2}\ \textbf{point}], \\ \operatorname{Arg} w &= \arctan\left(\frac{3}{3}\right) = \frac{\pi}{4} && [\tfrac{1}{2}\ \textbf{point}]. \end{aligned}$$

The principal argument should lie between $-\pi$ and π. For values of the principal argument that are correct up to an addition/subtraction of a multiple of 2π, subtract 1/2 point once.

(b) Using the expression for $|w|$ and $\operatorname{Arg}(w)$, if $v^2 = w$, then we have

$$\begin{aligned} |v| &= \sqrt{|w|} = \sqrt{3\sqrt{2}}, && [\tfrac{1}{2}\ \textbf{point}], \\ \arg(v) &= \tfrac{1}{2}\arg(w) = \tfrac{1}{2}(\operatorname{Arg}(w) + k2\pi) = \frac{\pi}{8} + k\pi, \quad k \in \mathbb{Z}, && [\tfrac{1}{2}\ \textbf{point}]. \end{aligned}$$

We thus find [**1 point**]

$$v = \sqrt{3\sqrt{2}}\,(\cos(\pi/8) + i\sin(\pi/8)) \quad \text{or} \quad v = \sqrt{3\sqrt{2}}\,(\cos(9\pi/8) + i\sin(9\pi/8))\,.$$

[If the solution is represented as

$$v = \sqrt{3\sqrt{2}}\,(\cos(\pi/8 + k\pi) + i\sin(\pi/8 + k\pi))\,, \quad k \in \mathbb{Z},$$

subtract $\frac{1}{2}$ point, as it should be clear that there are two solutions v only. It is fine if for example the angle $-7\pi/8$ is used instead of $9\pi/8$, since these correspond to the same v.]

6. (a) Using separation of variables, for $x > 0$,

$$\begin{aligned} x\frac{dy}{dx} &= (y+1)^2, \\ \int \frac{1}{(y+1)^2}\,dy &= \int \frac{1}{x}\,dx, && [\textbf{1 point}] \\ -\frac{1}{y+1} &= \ln(x) + c, \\ y(x) &= -\frac{1}{c + \ln x} - 1, && [\textbf{1 point}], \end{aligned}$$

for some constant $c \in \mathbb{R}$. [Forgetting the constant c: subtract 1 point.]

(b) The initial value problem $9y''(x) - 6y'(x) + 5y(x) = 0$ has characteristic (or auxiliary) equation

$$9r^2 - 6r + 5 = 0,$$

with roots $r = \frac{1}{3} + \frac{2}{3}i$ and $r = \frac{1}{3} - \frac{2}{3}$. [**1 point**] We see that the general solution is given by

$$y(x) = e^{\frac{1}{3}x}\left(C_1\cos(\tfrac{2}{3}x) + C_2\sin(\tfrac{2}{3}x)\right), \quad x \in \mathbb{R} \qquad [\textbf{1 point}].$$

The initial value $y(0) = 0$ yields $C_1 = 0$. We then have

$$y'(x) = \tfrac{1}{3}C_2 e^{\frac{1}{3}x} \sin(\tfrac{2}{3}x) + \tfrac{2}{3}C_2 e^{\frac{1}{3}x} \cos(\tfrac{2}{3}x),$$

so that $y'(0) = \frac{2}{3}C_2$. Putting $y'(0) = -2$ gives $C_2 = -3$, resulting in the solution

$$y(x) = -3e^{\frac{1}{3}x} \sin(\tfrac{2}{3}x), \quad x \in \mathbb{R}, \qquad \textbf{[1 point]}.$$

[Not giving the final expression for $y(x)$: subtract 1/2 point.]