First test Calculus 2, 18 November 2019, Solutions

Rough guideline:

- minor mistake (computation error): substract $\frac{1}{2}$ point;
- major mistake: substract 1 point

1. We have
$$\frac{-n}{n^2+1} \leq \frac{n \sin n}{n^2+1} \leq \frac{n}{n^2+1},$$
By the squeeze law we find $\lim_{n\to\infty} \frac{n\sin n}{n^2+1} = 0$, i.e. the sequence is convergent [**1 point**]. It follows that the sequence is bounded [**1 point**]. The sequence is not decreasing (there are infinitely many values n for which $\sin n < 0$ and $\sin(n+1) > 0$) and for the same reason the sequence is not increasing. Also the series is not alternating since $\sin n$ is not alternating ($\sin(\pi/2 + n\pi)$ would be alternating, but e.g. $\sin(5) < 0$ and $\sin(6) < 0$). [**1 point**]

2. (a) [**3 points**] We have $a_n = (-1)^n \frac{1}{\sqrt{n+1}+\sqrt{n}} = (-1)^n b_n$ with $b_n = \frac{1}{\sqrt{n+1}+\sqrt{n}} \to 0$ as $n \to \infty$. Also (b_n) is decreasing. By the alternating series test, $\sum_{n=1}^{\infty} a_n$ is convergent. The series is not absolutely convergent, since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} b_n = \infty$. ($b_n \geq \frac{1}{2\sqrt{n+1}}$ which up to the factor 1/2 gives a divergent p-series.) Therefore the series is *conditionally convergent*.

   (b) [**3 points**] We can use the ratio test. For $a_n = (-1)^n \frac{7n+(993)^n}{n!}$ we have
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{7(n+1)+(993)^{n+1}}{(n+1)!} \frac{n!}{7n+(993)^n} = \frac{7(n+1)+(993)^{n+1}}{(n+1)(7n+(993)^n)}$$
$$= \frac{7(n+1)(993)^{-n}+993}{(n+1)(7n(993)^{-n}+1)} \to 0$$
as $n \to \infty$. (The numerator converges to the constant 993, the large n behaviour of the denominator is $(n+1)$.) By the ratio test (where we have $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$) it follows that the series is *absolutely convergent*.

3. We may write the series $\sum_{n=1}^{\infty} \frac{(2-x)^n}{3^n n^{1/3}}$ as $\sum_{n=1}^{\infty} a_n (x-2)^n$, where
$$a_n = (-1)^n \frac{1}{3^n n^{1/3}}.$$
We have
$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3^n n^{1/3}}{3^{n+1}(n+1)^{1/3}} = \lim_{n\to\infty} \frac{1}{3}\left(\frac{n}{n+1}\right)^{1/3} = \frac{1}{3},$$

so that the radius of convergence is $R = 1/L = 3$. [**1 point**] The point $x = 5$ gives the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^{1/3}}$$

which is convergent using the alternating series test. [**1 point**] The point $x = -1$ gives the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$$

which is divergent since it is a p-series with $p = 1/3 \leq 1$. We conclude that the *interval of convergence* is $(-1, 5]$. [**1 point**]

4. At least two different approaches can be taken to determining the Taylor series; both give the correct solution and therefore score **2 points**. [It does not really matter how the resulting series is written, as long as the terms are correct for $n \geq 1$ and the series evaluates to $\frac{1}{2}$ for $x = 0$.]

   - Write

$$\begin{aligned} f(x) &= \frac{x}{1+x} = \frac{x+1}{1+x} - \frac{1}{1+x} = 1 - \frac{1}{2 + (x-1)} \\ &= 1 - \frac{1}{2}\left(\frac{1}{1 + (x-1)/2}\right) = 1 - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x-1}{2}\right)^n \\ &= \frac{1}{2} + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2^{n+1}}(x-1)^n. \end{aligned}$$

   - For $f(x) = \frac{x}{1+x}$ we have

$$f'(x) = \frac{1}{1+x} - \frac{x}{(1+x)^2} = \frac{1}{(1+x)^2}, \quad f^{(2)}(x) = \frac{(-2)}{(1+x)^3},$$

   and by induction

$$f^{(n)}(x) = (-1)^{n+1}\frac{n!}{(1+x)^{n+1}} \quad \text{for all } n \geq 1,$$

   so that $f^{(n)}(1) = (-1)^{n+1}\frac{n!}{2^{n+1}}$ for all $n \geq 1$. The resulting Taylor series is

$$\sum_{n=0}^{\infty} f^{(n)}(1)\frac{(x-1)^n}{n!} = \frac{1}{2} + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2^{n+1}}(x-1)^n.$$

We still have to determine the radius of convergence [**1 point**]. We have for $a_n = (-1)^{n+1}\frac{1}{2^{n+1}}$ that

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{n+1}}{2^{n+2}} = \frac{1}{2},$$

so that by the ratio test the *radius of convergence* is $R = 1/L = 2$.

5. (a) We have [**1 point**]

$$\mathbf{u} \bullet \mathbf{v} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 4 + 2 - 2 = 4,$$

and [**1 point**]

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 - (-2) \cdot 2 \\ -(4 \cdot 1 - (-2) \cdot 1) \\ 4 \cdot 2 - 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -6 \\ 7 \end{pmatrix} = 5\mathbf{i} - 6\mathbf{j} + 7\mathbf{k}.$$

(b) We have [**2 points**]

$$\mathbf{u}_{\mathbf{v}} = \left(\frac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} = \left(\frac{4}{1^2 + 2^2 + 1^2} \right) \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{4}{3} \\ \frac{2}{3} \end{pmatrix} = \tfrac{2}{3}\mathbf{i} + \tfrac{4}{3}\mathbf{j} + \tfrac{2}{3}\mathbf{k}.$$

(c) [**2 points**] This plane is given by

$$4(x-2) + (y-1) - 2(z-4) = 0$$

(which can be read off directly from $\mathbf{u} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ and the point $P = (2, 1, 4)$)

or equivalently

$$4x + y - 2z = 1.$$

(d) The line L has parametrization $\mathbf{r}(t) = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2+t \\ 1+2t \\ 4+t \end{pmatrix}$ [**1 point**].

Different solution methods, all good for **1 point**:

- The distance from the origin to $\mathbf{r}(t)$ is given for $t \in \mathbb{R}$ by

  $$|\mathbf{r}(t)| = \sqrt{(2+t)^2 + (1+2t)^2 + (4+t)^2} = \sqrt{6t^2 + 16t + 21}.$$

  We have

  $$|\mathbf{r}(t)|^2 = 6t^2 + 16t + 21 = 6(t + \tfrac{4}{3})^2 + \frac{31}{3}$$

  which is minized at $t_0 = -\frac{4}{3}$, giving

  $$|\mathbf{r}(t_0)| = \sqrt{\frac{31}{3}} = \frac{1}{3}\sqrt{93}.$$

- The distance is minimized when the position vector $\mathbf{r}(t)$ is orthogonal to the direction $\mathbf{v}$, i.e. we require

  $$0 = \mathbf{v} \bullet \mathbf{r}(t) = (2+t) + 2(1+2t) + 1(4+t) = 8 + 6t,$$

  which is satisfied at $t_0 = -\frac{4}{3}$. We still need to compute the distance at $t = t_0$, using the formula for $|\mathbf{r}(t)|$ given above.

- One might remember the formula

$$\left|\mathbf{r}_0 - \mathbf{r}_1 - \frac{\mathbf{v} \bullet (\mathbf{r}_0 - \mathbf{r}_1)}{|\mathbf{v}|^2}\mathbf{v}\right|$$

  from the lectures, or

$$\frac{|(\mathbf{r}_0 - \mathbf{r}_1) \times \mathbf{v}|}{|\mathbf{v}|},$$

  from the book, which could be applied for

$$\mathbf{r}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{r}_1 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},$$

  giving the same answer $\sqrt{\frac{31}{3}} = \frac{1}{3}\sqrt{93}$.

6. $f(x, y) = \frac{\cos(xy)}{1-y^2}$.

   (a) Each correct partial derivative scores **1 point**:

$$\frac{\partial f}{\partial x}(x, y) = -\frac{y\sin(xy)}{1-y^2}, \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = -\frac{x\sin(xy)}{1-y^2} + \frac{2y\cos(xy)}{(1-y^2)^2}.$$

   (b) [**2 points**] The tangent plane is then given, for $(x_0, y_0) = (1/2, \pi)$ by

$$\begin{aligned} z &= f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) \\ &= 0 - \frac{\pi}{1-\pi^2}(x - 1/2) - \frac{1}{2(1-\pi^2)}(y - \pi) \end{aligned}$$

   or equivalently

$$\left(\frac{\pi}{\pi^2 - 1}\right)x + \left(\frac{1}{2(\pi^2 - 1)}\right)y - z = \frac{\pi}{\pi^2 - 1}.$$

   (For other expressions, check if the plane has normal $\begin{pmatrix} \pi/(\pi^2 - 1) \\ 1/(2(\pi^2 - 1)) \\ -1 \end{pmatrix}$ and $P_0 = (x_0, y_0, z_0) = (1/2, \pi, 0)$ lies on the plane.)