Resit Calculus 2, 7 February 2019, Solutions

1. a) First consider $\sum\limits_{n=1}^{\infty} \left|\frac{\sin(n)}{\sqrt{1+n^3}}\right|$. Since $|\sin(n)| < 1$ we have

$$\left|\frac{\sin(n)}{\sqrt{1+n^3}}\right| \leq \frac{1}{\sqrt{1+n^3}} < \frac{1}{n^{3/2}}$$

and since $\sum\limits_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (p-series with $p = 3/2 > 1$), the series $\sum\limits_{n=1}^{\infty} \frac{\sin(n)}{\sqrt{n^3+1}}$ converges absolutely (comparison test).

b) First consider

$$\sum_{n=1}^{\infty}\left|(-1)^n \frac{\arctan(n)}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{\arctan(n)}{\sqrt{n}}.$$

We will use the limit comparison test:

$$\lim_{n\to\infty} \frac{\arctan(n)}{\sqrt{n}} \div \frac{1}{\sqrt{n}} = \frac{\pi}{2}.$$

And because $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series with $p = 1/2 < 1$), the series $\sum\limits_{n=1}^{\infty} \frac{\arctan(n)}{\sqrt{n}}$ diverges also. This means that $\sum\limits_{n=1}^{\infty} (-1)^n \frac{\arctan(n)}{\sqrt{n}}$ is not absolutely convergent.
Now use the alternating series test:
(1) The series is clearly alternating,
(2) The sequence $\left\{\frac{\arctan(n)}{\sqrt{n}}\right\}$ is decreasing (introduce $f(x) = \frac{\arctan(x)}{\sqrt{x}}$, then $f'(x) = \frac{2x-(1+x^2)\arctan(x)}{2x\sqrt{x}(1+x^2)} < 0$ for large x, so f is ultimately decreasing), and
(3) The sequence $\left\{\frac{\arctan(n)}{\sqrt{n}}\right\}$ has limit 0.
So the series $\sum\limits_{n=1}^{\infty} (-1)^n \frac{\arctan(n)}{\sqrt{n}}$ is convergent.
Finally we can conclude that the series is conditionally convergent.

2. We use the ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(2x-5)^{2(n+1)}}{(n+1)^4\, 9^{n+1}}\right| \div \left|\frac{(2x-5)^{2n}}{n^4\, 9^n}\right| \\ &= \frac{|2x-5|^2}{9} \lim_{n\to\infty} \frac{n^4}{(n+1)^4} = \frac{|2x-5|^2}{9}.\end{aligned}$$

So the series converges absolutely for $|2x-5|^2 < 9$, that is for $1 < x < 4$, and diverges for $|2x-5|^2 > 9$.
Now determine the behavior in the endpoints: For both $x = 1$ and $x = 4$ we find the p-series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ which is convergent. So the interval of convergence is $[1, 4]$.

3. a) Use the Maclaurin-series representation for the exponential function:

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \text{ which converges for all } t \in \mathbb{R}.$$

This yields (substitute $t = x^2$ and $t = -x^2$)

$$\frac{1}{2}\left(e^{x^2} - e^{-x^2}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^{2n}}{n!} - \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!} = \frac{1}{2}\sum_{n=0}^{\infty}(1-(-1)^n)\frac{x^{2n}}{n!}.$$

Now $1-(-1)^n = 0$ for even n, and $1-(-1)^n = 2$ for odd n. This results in

$$f(x) = \frac{1}{2}\sum_{n=0}^{\infty}(1-(-1)^{2n+1})\frac{x^{2(2n+1)}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{x^{4n+2}}{(2n+1)!}.$$

b) Since the Maclaurin-series converges for all $x \in \mathbb{R}$ we may use term-by-term differentiation. This yields

$$f'(x) = \sum_{n=0}^{\infty}\frac{d}{dx}\frac{x^{4n+2}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{4n+2}{(2n+1)!}x^{4n+1} = \sum_{n=0}^{\infty}2\frac{1}{(2n)!}x^{4n+1}.$$

On the other hand $f'(x) = \frac{1}{2}\left(2xe^{x^2} + 2xe^{-x^2}\right) = x(e^{x^2}+e^{-x^2})$. Now substitute $x = 1$ in both expressions, and divide by 2, to obtain

$$\sum_{n=0}^{\infty}\frac{1}{(2n)!} = \frac{e+e^{-1}}{2}.$$

4. a) $\mathbf{u} \bullet \mathbf{v} = 0 + 0 - 2 = -2$ and $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 2 \\ -6 \\ 6 \end{pmatrix} = 2\mathbf{i} - 6\mathbf{j} + 6\mathbf{k}$.

b) $\mathbf{u_v} = \dfrac{\mathbf{u} \bullet \mathbf{v}}{\mathbf{v} \bullet \mathbf{v}}\mathbf{v} = \dfrac{-2}{8}\mathbf{v} = \begin{pmatrix} 0 \\ -1/2 \\ -1/2 \end{pmatrix} = -\dfrac{1}{2}\mathbf{j} - \dfrac{1}{2}\mathbf{k}$.

c) The equation is:

$$3(x-1) + 0(y-2) - 1(z-3) = 0, \quad \text{or equivalently } \ 3x - z = 0.$$

5. a) Calculate both first partial derivatives and set them equal to 0:

$$f_x(x,y) = 0 \Longrightarrow 6x^2 - 30 + 6y = 0 \Longrightarrow y = 5 - x^2.$$

$$f_y(x,y) = 0 \Longrightarrow 6x + 6y + 6 = 0 \Longrightarrow y = -1 - x.$$

It follows that $5 - x^2 = -1 - x$, so $x^2 - x - 6 = (x-3)(x+2) = 0$, with solutions $x = 3$ and $x = -2$. Therefore we find two critical points: $S_1 = (3, -4)$ and $S_2 = (-2, 1)$.

b) For general (x,y) we find $f_{xx}(x,y) = 12x, f_{yy}(x,y) = 6$ and $f_{xy}(x,y) = 6 = f_{yx}(x,y)$. So we find $f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)f_{yx}(x,y) = 36(2x-1)$. This implies that S_2 is a saddle point ($f_{xx}f_{yy} - f_{xy}f_{yx} < 0$) and that f has a local minimum value in S_1 ($f_{xx}f_{yy} - f_{xy}f_{yx} > 0$ and $f_{xx} > 0$).

c) The unit vector $\mathbf{v}$ in the same direction as $\mathbf{u}$ is given by

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix}. \text{ Furthermore } \nabla f(1,2) = \begin{pmatrix} f_x(1,2) \\ f_y(1,2) \end{pmatrix} = \begin{pmatrix} -12 \\ 24 \end{pmatrix}.$$

So $D_{\mathbf{v}}(1,2) = \mathbf{v} \bullet \nabla f(1,2) = -\frac{132}{5}$.

6. a) Make a sketch of the domain. Then you can easily verify that

$$\int_0^1 \int_{\sqrt{x}}^1 \ln(1+y^3)\, dy\, dx = \int_0^1 \int_0^{y^2} \ln(1+y^3)\, dx\, dy$$

$$= \int_0^1 \ln(1+y^3) \Big[x\Big]_{x=0}^{x=y^2} dy = \int_0^1 y^2 \ln(1+y^3)\, dy$$

$$\overset{(*)}{=} \int_1^2 \frac{1}{3} \ln(t)\, dt \overset{(**)}{=} \frac{1}{3}\Big[t\ln(t) - t\Big]_{t=1}^{t=2} = \frac{1}{3}(2\ln(2) - 1).$$

At $(*)$ we used the substitution $t = 1 + y^3$ and at $(**)$ we used integration by parts.

b) Again sketch the domain. It is that part of the disk around $(0,0)$ with radius $\sqrt{5}$, under the line $y = 0$. Using $x = r\cos(\theta), y = r\sin(\theta)$ we get

$$\int\int_S e^{-x^2-y^2}\, dA = \int_0^{\sqrt{5}} \int_{\pi}^{2\pi} re^{-r^2}\, d\theta\, dr =$$

$$= \pi \int_0^{\sqrt{5}} re^{-r^2}\, dr = \pi\Big[-\frac{1}{2}e^{-r^2}\Big]_0^{\sqrt{5}} = \frac{\pi}{2}\left(1 - e^{-5}\right).$$

7. Suppose $w = 2 + 2i$ and $z = \sqrt{3} + i$. Then $|w| = \sqrt{8} = 2\sqrt{2}$, $\arg(w) = \frac{\pi}{4}$, $|z| = 2$ and $\arg(z) = \frac{\pi}{6}$. Then

$$\left|\frac{w^3}{z^4}\right| = \frac{|w|^3}{|z|^4} = \frac{16\sqrt{2}}{16} = \sqrt{2}$$

and

$$\arg\left(\frac{w^3}{z^4}\right) = 3\arg(w) - 4\arg(z) = \frac{1}{12}\pi.$$

Therefore

$$\frac{(2+2i)^3}{(\sqrt{3}+i)^4} = \sqrt{2}\left(\cos\left(\frac{1}{12}\pi\right) + i\sin\left(\frac{1}{12}\pi\right)\right),$$

so $a = \sqrt{2}\cos(\frac{1}{12}\pi)$ en $b = \sqrt{2}\sin(\frac{1}{12}\pi)$. [This may also be written as $a = \frac{1}{2} + \frac{1}{2}\sqrt{3}$ and $b = -\frac{1}{2} + \frac{1}{2}\sqrt{3}$, which would be the result of a slightly different calculation. Both answers are correct.]

8. This is a linear differential equation of order one. First divide both sides by x^2 to get

$$y'(x) - \frac{1}{x^2}\, y(x) = \frac{1}{x^2}.$$

Now remark that an integrating factor is $e^{\int -\frac{1}{x^2}\,dx} = e^{\frac{1}{x}}$. This yields

$$\frac{d}{dx}\left(y(x)e^{\frac{1}{x}}\right) = \frac{1}{x^2}e^{\frac{1}{x}} \Longrightarrow y(x)e^{\frac{1}{x}} = \int \frac{1}{x^2}e^{\frac{1}{x}}\,dx = -e^{\frac{1}{x}} + C.$$

[You can use the substitution $t = \frac{1}{x}$ if necessary.] So the general solution is

$$y(x) = -1 + Ce^{-\frac{1}{x}}, C \in \mathbb{R}.$$

Substitution of the initial value gives

$$2 = y(1) = -1 + Ce^{-1}, \qquad \text{so} \qquad C = 3e.$$

The solution of the initial value problem is therefore $y(x) = -1 + 3e^{1-\frac{1}{x}}$.
[This problem can also be solved using separation of the variables!]