Second test Calculus 2, 18 December 2018, Solutions

1. First
$$\frac{\partial}{\partial y} f(2x-3y, 4xy) = -3f_1(2x-3y, 4xy) + 4xf_2(2x-3y, 4xy).$$
Then (to shorten the formulas somewhat we write $u = 2x - 3y$ and $v = 4xy$).
$$\frac{\partial^2}{\partial x \partial y} f(u,v) = \frac{\partial}{\partial x} \left(-3f_1(u,v) + 4xf_2(u,v)\right) =$$
$$= -3\left(2f_{11}(u,v) + 4yf_{12}(u,v)\right) + 4f_2(u,v) + 4x\left(2f_{21}(u,v) + 4yf_{22}(u,v)\right) =$$
$$= 4f_2(u,v) - 6f_{11}(u,v) - 12yf_{12}(u,v) + 8xf_{21}(u,v) + 16xyf_{22}(u,v).$$

2. a) Calculate both first partial derivatives and set them equal to 0:
$$f_x(x,y) = 0 \Longrightarrow y^2 - 2xy = 0.$$
$$f_y(x,y) = 0 \Longrightarrow 2xy - x^2 - 3y^2 + 4 = 0.$$
The first equation yields $y = 0$ (case I) or $y = 2x$ (case II). Case I: substitution of $y = 0$ in the second equation gives $x^2 = 4$, so $x = 2$ or $x = -2$. So we find two critical points $S_1 = (2,0)$ and $S_2 = (-2,0)$. Case II: substitution of $y = 2x$ in the second equation yields $9x^2 = 4$, so $x = \frac{2}{3}$ or $x = -\frac{2}{3}$. So we find two more critical points $S_3 = (\frac{2}{3}, \frac{4}{3})$ and $S_4 = (-\frac{2}{3}, -\frac{4}{3})$.

   b) For general (x,y) we find
$$f_{xx}(x,y) = -2y,\ \ f_{yy}(x,y) = 2x - 6y \quad \text{and} \quad f_{xy}(x,y) = 2y - 2x = f_{yx}(x,y).$$
So the determinant of the Hesse matrix is
$$f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)f_{yx}(x,y) = 4(2y^2 + xy - x^2).$$
This implies that S_1 and S_2 are saddle points ($f_{xx}f_{yy} - f_{xy}f_{yx} = -16 < 0$) and that f has a local maximum value $\frac{32}{9}$ in S_3 ($f_{xx}f_{yy} - f_{xy}f_{yx} = 16 > 0$ and $f_{xx} = -\frac{8}{3} < 0$) and a local minimum value $-\frac{32}{9}$ in S_4 ($f_{xx}f_{yy} - f_{xy}f_{yx} = 16 > 0$ and $f_{xx} = \frac{8}{3} > 0$).

   c) The extreme values $\frac{32}{9}$ and $-\frac{32}{9}$ found in 2b) are local values. This can be shown by investigating $f(0,y) = -y^3 + 4y$. Clearly f tends to $-\infty$ when $y \to \infty$ and f tends to ∞ when $y \to -\infty$.

3. Introduce the Lagrange function $L(x,y,\lambda) = xy + \lambda(4x^2 + y^2 - 8)$ and find its critical points:
$$\begin{cases} 0 = \frac{\partial L}{\partial x} = y + 8\lambda x & (A) \\ 0 = \frac{\partial L}{\partial y} = x + 2\lambda y & (B) \\ 0 = \frac{\partial L}{\partial \lambda} = 4x^2 + y^2 - 8 & (C) \end{cases}$$
Multiply equation (A) with y and multiply equation (B) with $4x$. Then subtract one from the other to eliminate λ and to obtain $y^2 = 4x^2$. Substitute this result in equation (C) to obtain $x^2 = 1$, so $x = 1$ or $x = -1$. So we find four critical points: $S_1 = (1,2), S_2 = (1,-2), S_3 = (-1,2)$ and $S_4 = (-1,-2)$. Now calculate $f(S_1) = f(S_4) = 2$ (maximum value) and $f(S_2) = f(S_3) = -2$ (minimum value).

4. a) Since we cannot find an antiderivative of $\frac{x}{1+x^5}$ easily we will reverse the order of integration. Make a sketch of the domain and find:

$$\int_0^1 \int_{\sqrt{y}}^1 \frac{x\sqrt{y}}{1+x^5}\, dx\, dy = \int_0^1 \int_0^{x^2} \frac{x\sqrt{y}}{1+x^5}\, dy\, dx = \int_0^1 \frac{x}{1+x^5}\left[\frac{2}{3}y\sqrt{y}\right]_{y=0}^{y=x^2} dx$$

$$= \int_0^1 \frac{2}{3}\frac{x^4}{1+x^5}\, dx = \left[\frac{2}{15}\ln\left(1+x^5\right)\right]_{x=0}^{x=1} = \frac{2}{15}\ln(2).$$

b) Again sketch the domain. It is the part of the (x, y)-plane between two circles with center $(0, 0)$ and radius 1 resp. $\sqrt{2}$, under the line $y = x$ and above the line $y = -x$. Using polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$ we get

$$\int\int_S x\sqrt{x^2+y^2}\, dA = \int_{-\pi/4}^{\pi/4}\int_1^{\sqrt{2}} r^3\cos(\theta)\, dr\, d\theta =$$

$$= \int_{-\pi/4}^{\pi/4}\left[\frac{1}{4}r^4\right]_{r=1}^{r=\sqrt{2}}\cos(\theta)\, d\theta = \int_{-\pi/4}^{\pi/4}\frac{3}{4}\cos(\theta)\, d\theta = \left[\frac{3}{4}\sin(\theta)\right]_{\theta=-\pi/4}^{\pi/4} = \frac{3}{4}\sqrt{2}.$$

5. a) Multiply numerator and denominator with the complex conjugate of the denominator:

$$z = \frac{4-i}{3-2i} \times \frac{3+2i}{3+2i} = \frac{12+5i-2i^2}{9-4i^2} = \frac{14+5i}{13},$$

so the real part of z is $\frac{14}{13}$ and the imaginary part of z is $\frac{5}{13}$.

b) We have

$$|z^2| = |\sqrt{3}-i| = 2 \qquad \text{and} \qquad \arg(z^2) = \arg(\sqrt{3}-i) + 2k\pi = -\frac{\pi}{6} + 2k\pi.$$

Since $|z^2| = |z|^2$, the first equation yields $|z| = \sqrt{2}$. And since $\arg(z^2) = 2\arg(z)$ the second equation gives $\arg(z) = -\frac{\pi}{12} + k\pi$. So the two solutions are (choose $k = 0, 1$):

$$z_1 = \sqrt{2}\left(\cos\left(-\frac{1}{12}\pi\right) + i\sin\left(-\frac{1}{12}\pi\right)\right),$$

$$z_2 = \sqrt{2}\left(\cos\left(\frac{11}{12}\pi\right) + i\sin\left(\frac{11}{12}\pi\right)\right).$$

6. First multiply the equation by r, which leads to $r^2 = 2r\sin\theta + 4r\cos\theta$. Then the transformation from polar to rectangular coordinates ($x = r\cos\theta$ and $y = r\sin\theta$) gives $x^2 + y^2 = 2y + 4x$. This can be rewritten as $(x-2)^2 + (y-1)^2 = 5$, a circle with center $(2, 1)$ and radius $\sqrt{5}$.

7. This is a first order differential equation that is separable. Furthermore it is clear that $y \equiv 0$ (so $y(x) = 0$ for all x) is a solution. Now assume $y \not\equiv 0$ and separate the variables to find (you may substitute $1 + x^2 = t$ in the second integral):

$$\int \frac{1}{\sqrt{y}}\, dy = \int \frac{x}{1+x^2}\, dx \Longrightarrow 2\sqrt{y} = \frac{1}{2}\ln\left(1+x^2\right) + C,$$

so the general explicit solutions are $y(x) = \frac{1}{16}\left(\ln\left(1+x^2\right) + 2C\right)^2$, $C \in \mathbb{R}$ and $y \equiv 0$.

8. Substitute $y(x) = e^{rx}$. Then the auxiliary equation becomes

$$r^2 + 2r + 5 = (r + 1)^2 + 4 = 0,$$

with two complex solutions $r = -1 + 2i$ and $r = -1 - 2i$. So the general real solution is:

$$y(x) = c_1 e^{-x} \cos(2x) + c_2 e^{-x} \sin(2x), c_1, c_2 \in \mathbb{R}.$$

Substitution of the first initial value condition yields $3 = y(0) = c_1$. Now differentiate the general solution to get

$$y'(x) = -c_1 e^{-x} \cos(2x) - 2c_1 e^{-x} \sin(2x) - c_2 e^{-x} \sin(2x) + 2c_2 e^{-x} \cos(2x).$$

Finally substitute the second initial value condition (and also use $c_1 = 3$), to get

$$-3 = y'(0) = -c_1 + 2c_2 = -3 + 2c_2, \text{ so } c_2 = 0.$$

So the solution of this initial value problem is $y(x) = 3e^{-x} \cos(2x)$.