First test Calculus 2, 19 November 2018, Solutions

1. First remark that $\cos(n\pi) = (-1)^n$ for all $n \in \mathbb{N}$ and that the denominator is always positive. Therefore the sequence is alternating. Furthermore:

$$-\frac{1}{\sqrt{n+1}} \leq \frac{(-1)^n}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n+1}}.$$

So with the squeeze law we find $\lim\limits_{n\to\infty} \frac{(-1)^n}{\sqrt{n+1}} = 0$, which implies that the sequence is convergent (with limit 0). As a consequence the sequence is bounded (for example bounded above by its maximum $\frac{1}{\sqrt{3}}$ and bounded below by its minimum $-\frac{1}{\sqrt{2}}$).

2. a) **[2 points]** Use the ratio test and the fact that $(n+1)! = (n+1)n!$ to find that

$$\lim_{n\to\infty} \left| (-1)^{n+1}\frac{n+1+4^{n+1}}{(n+1)!} \div (-1)^n \frac{n+4^n}{n!} \right| = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{\frac{n+1}{4^n}+4}{\frac{n}{4^n}+1} = 0 \cdot 4 = 0,$$

since "4^n wins from n" (or use l'Hospital). So the series is absolutely convergent.

b) **[4 points]** First consider $\sum\limits_{n=2}^{\infty} \left| \frac{(-1)^n \ln(n)}{\sqrt{n}} \right| = \sum\limits_{n=2}^{\infty} \frac{\ln(n)}{\sqrt{n}}$. Since $\frac{\ln(n)}{\sqrt{(n)}} > \frac{1}{\sqrt{n}}$ for all $n \geq 3$ and since $\sum\limits_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series with $p = \frac{1}{2}$), the series $\sum\limits_{n=2}^{\infty} \frac{\ln(n)}{\sqrt{n}}$ also diverges (comparison test). So $\sum\limits_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{\sqrt{n}}$ is not absolutely convergent.

Now apply the alternating series test to $\sum\limits_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{\sqrt{n}}$:

(i) the series is clearly alternating,

(ii) the sequence $\left\{ \frac{\ln(n)}{\sqrt{n}} \right\}$ is ultimately decreasing (since $f(x) = \frac{\ln(x)}{\sqrt{x}}$ implies $f'(x) = \frac{2-\ln(x)}{2x\sqrt{x}} < 0$ for $x > e^2$, so f is ultimately decreasing),

(iii) the sequence $\left\{ \frac{\ln(n)}{\sqrt{n}} \right\}$ has limit 0 ("$\sqrt{n}$ wins from $\ln(n)$" or use l'Hospital).

So the series $\sum\limits_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{\sqrt{n}}$ is convergent.

Finally we can conclude that the series is conditionally convergent.

3. a) We use the ratio test:

$$\lim_{n\to\infty} \left| \frac{(2x+3)^{n+1}}{(n+1)n4^{n+1}} \div \frac{(2x+3)^n}{n(n-1)4^n} \right| = \frac{|2x+3|}{4} \lim_{n\to\infty} \frac{(n-1)}{(n+1)} = \frac{|2x+3|}{4}.$$

So the series converges absolutely for $|2x+3| < 4$, that is for $-\frac{7}{2} < x < \frac{1}{2}$, and diverges for $|2x+3| > 4$, which is for $x < -\frac{7}{2}$ or for $x > \frac{1}{2}$.

Now determine separately the behavior in the endpoints: First substitute $x = \frac{1}{2}$ to find $\sum\limits_{n=1}^{\infty} \frac{1}{n(n-1)}$ which is an absolutely convergent series (use for example the limit comparison test and compare with the convergent p-series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$).

Then substitute $x = -\frac{7}{2}$ to get $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n(n-1)}$. And this series is also absolutely convergent (compare with the previous series). So the interval of convergence is $[-\frac{7}{2}, \frac{1}{2}]$.

b) On $(-\frac{7}{2}, \frac{1}{2})$ we may differentiate termwise twice, to obtain:

$$f'(x) = \sum_{n=2}^{\infty} \frac{2(2x+3)^{n-1}}{(n-1)4^n} \Longrightarrow f''(x) = \sum_{n=2}^{\infty} \frac{4(2x+3)^{n-2}}{4^n}.$$

Therefore (geometric series!):

$$f''(0) = \sum_{n=2}^{\infty} \frac{4(3)^{n-2}}{4^n} = \frac{1}{4}\sum_{n=2}^{\infty} \frac{3^{n-2}}{4^{n-2}} = \frac{1}{4}\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{1}{4} \cdot \frac{1}{1-\frac{3}{4}} = 1.$$

4. We use the geometric series

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n, \text{ converging for all } |t| < 1.$$

Then

$$\frac{2x}{3+x^2} = \frac{2x}{3} \cdot \frac{1}{1-(-\frac{x^2}{3})} = \frac{2x}{3}\sum_{n=0}^{\infty} \left(-\frac{x^2}{3}\right)^n = \sum_{n=0}^{\infty} \frac{2(-1)^n}{3^{n+1}} x^{2n+1},$$

which converges for $|-\frac{x^2}{3}| < 1$, so for all $|x| < \sqrt{3}$.

5. a) $\mathbf{u} \bullet \mathbf{v} = 2 - 3 + 2 = 1$ and $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -7\mathbf{i} - 3\mathbf{j} + 5\mathbf{k} = \begin{pmatrix} -7 \\ -3 \\ 5 \end{pmatrix}$.

b) The required vectors should be parallel to $\mathbf{u} \times \mathbf{v}$, and have length 1. Since $|\mathbf{u} \times \mathbf{v}| = \sqrt{49+9+25} = \sqrt{83}$ we find the two vectors

$$\frac{1}{\sqrt{83}}\begin{pmatrix} -7 \\ -3 \\ 5 \end{pmatrix} = -\frac{7}{\sqrt{83}}\mathbf{i} - \frac{3}{\sqrt{83}}\mathbf{j} + \frac{5}{\sqrt{83}}\mathbf{k},$$

and

$$-\frac{1}{\sqrt{83}}\begin{pmatrix} -7 \\ -3 \\ 5 \end{pmatrix} = \frac{7}{\sqrt{83}}\mathbf{i} + \frac{3}{\sqrt{83}}\mathbf{j} - \frac{5}{\sqrt{83}}\mathbf{k}.$$

c) $\mathbf{u_v} = \dfrac{\mathbf{u} \bullet \mathbf{v}}{\mathbf{v} \bullet \mathbf{v}}\mathbf{v} = \dfrac{1}{6}\mathbf{v} = \begin{pmatrix} 1/6 \\ 1/6 \\ 1/3 \end{pmatrix} = \dfrac{1}{6}\mathbf{i} + \dfrac{1}{6}\mathbf{j} + \dfrac{1}{3}\mathbf{k}.$

6. A normal vector of the plane is perpendicular to all vectors in that plane. Two of these vectors are:

$$\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} = -\mathbf{i} - 2\mathbf{j} \text{ and } \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}.$$

So a normal vector is given by:

$$\begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} = -2\mathbf{i} + \mathbf{j} + 5\mathbf{k}.$$

So an equation of the plane is $-2(x-1) + 1(y-0) + 5(z-3) = 0$, or equivalently $-2x + y + 5z = 13$.

7. a) The partial derivatives are $f_x(x,y) = \dfrac{xy^4}{\sqrt{12+x^2y^4}}$ and $f_y(x,y) = \dfrac{2x^2y^3}{\sqrt{12+x^2y^4}}$.

b) (i) First remark that $f(2,1) = 4$. Furthermore: $f_x(2,1) = \frac{1}{2}$ and $f_y(2,1) = 2$. So an equation of the tangent plane is $z = 4+\frac{1}{2}(x-2)+2(y-1) = 1+\frac{1}{2}x+2y$.

(ii) The normal line to the graph of f in $(x,y,z) = (2,1,4)$ has direction vector

$$\mathbf{v} = f_x(2,1)\mathbf{i} + f_y(2,1)\mathbf{j} - \mathbf{k} = \tfrac{1}{2}\mathbf{i} + 2\mathbf{j} - \mathbf{k},$$

so in vector parametric notation the equation of the normal line is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} \Longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1/2 \\ 2 \\ -1 \end{pmatrix}, t \in \mathbb{R}.$$

In scalar parametric notation this is

$$\begin{cases} x = 2 + \frac{1}{2}t \\ y = 1 + 2t \\ z = 4 - t \end{cases}, t \in \mathbb{R}.$$

And in standard form this is

$$\frac{x-2}{1/2} = \frac{y-1}{2} = \frac{z-4}{-1}.$$