Resit Calculus 2, 5 April 2018, Solutions

1. a) **(3 points)** First consider $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n\arctan(n)}\right| = \sum_{n=1}^{\infty}\frac{1}{n\arctan(n)}$. Since $\lim_{n\to\infty}\arctan(n) = \frac{\pi}{2}$ we can compare the general term with $\frac{1}{n}$. This yields $\lim_{n\to\infty}\frac{1}{n\arctan(n)} : \frac{1}{n} = \frac{2}{\pi}$ and since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (p-series with $p=1$), the series $\sum_{n=1}^{\infty}\frac{1}{n\arctan(n)}$ also diverges (limit comparison test). So $\sum_{n=1}^{\infty}\frac{(-1)^n}{n\arctan(n)}$ is not absolutely convergent. Now use the alternating series test: (1) the series is alternating, (2) the sequence $\left\{\frac{1}{n\arctan(n)}\right\}$ is decreasing (since n and $\arctan(n)$ are both increasing), and (3) the sequence $\left\{\frac{1}{n\arctan(n)}\right\}$ has limit 0. So the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n\arctan(n)}$ is convergent. Finally we can conclude that the series is conditionally convergent.

   b) **(1 point)** Since $\lim_{n\to\infty}\sqrt{1+\frac{1}{n^2}} = 1$ it is clear that the general term of this series does not converge to 0. Therefore the series is divergent (general term test).

2. We use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+3)^{2(n+1)}}{(n+1)^2 4^{n+1}}\right| \div \left|\frac{(x+3)^{2n}}{n^2 4^n}\right|$$

$$= \frac{|x+3|^2}{4}\lim_{n\to\infty}\frac{n^2}{(n+1)^2} = \frac{|x+3|^2}{4}.$$

So the series converges absolutely for $|x+3|^2 < 4$, that is for $-5 < x < -1$, and diverges for $|x+3|^2 > 4$. Now determine the behavior in the endpoints: For both $x=-5$ and $x=-1$ we find the p-series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ which is convergent. So the interval of convergence is $[-5,-1]$.

3. a) Use the geometric series $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$, which converges for all $t\in(-1,1)$. This yields (substitute $t=-x^4$)

$$\frac{2x}{1+x^4} = 2x\cdot\frac{1}{1-(-x^4)} = 2x\sum_{n=0}^{\infty}\left(-x^4\right)^n = \sum_{n=0}^{\infty}2(-1)^n x^{4n+1},$$

   converging for $|-x^4| < 1$, so for $|x| < 1$.

   b) See part a). So the representation is valid for all $x\in(-1,1)$.

   c) Remark that $\frac{d}{dx}\arctan(x^2) = \frac{2x}{1+x^4}$, which is exactly the function considered in part a). So we can find the Maclaurin series-representation for $\arctan(x^2)$ by termwise integration of the representation given in part a). So on $(-1,1)$ we have

$$\arctan(x^2) = \int_0^x \sum_{n=0}^{\infty}2(-1)^n x^{4n+1}\,dx = \sum_{n=0}^{\infty}2(-1)^n\int_0^x x^{4n+1}\,dx$$

$$= \sum_{n=0}^{\infty}\frac{2(-1)^n}{4n+2}x^{4n+2}\Big|_0^x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{4n+2}.$$

4. a) A normal vector of the plane is perpendicular to all vectors in that plane. Two of these vectors are:

$$\begin{pmatrix}1\\-1\\1\end{pmatrix} - \begin{pmatrix}0\\2\\3\end{pmatrix} = \begin{pmatrix}1\\-3\\-2\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}1\\-1\\1\end{pmatrix} - \begin{pmatrix}-2\\0\\-1\end{pmatrix} = \begin{pmatrix}3\\-1\\2\end{pmatrix}.$$

So a normal vector is given by:

$$\begin{pmatrix}1\\-3\\-2\end{pmatrix} \times \begin{pmatrix}3\\-1\\2\end{pmatrix} = \begin{pmatrix}-8\\-8\\8\end{pmatrix}.$$

So an equation of the plane is $-8(x-1)-8(y+1)+8(z-1)=0$, or equivalently $x+y-z+1=0$.

b) The distance from the point $(2,2,1)$ to this plane is:

$$\frac{|2\cdot 1+2\cdot 1+1\cdot(-1)-(-1)|}{\sqrt{1^2+1^2+(-1)^2}} = \frac{4}{\sqrt{3}} = \frac{4}{3}\sqrt{3}.$$

5. Use the chain rule:

$$\frac{\partial}{\partial x} f(xe^y, e^x y^2) = e^y f_1(xe^y, e^x y^2) + e^x y^2 f_2(xe^y, e^x y^2)$$

and

$$\frac{\partial}{\partial y} f(xe^y, e^x y^2) = xe^y f_1(xe^y, e^x y^2) + 2e^x y f_2(xe^y, e^x y^2).$$

6. a) Calculate both first partial derivatives and set them equal to 0:

$$f_x(x,y)=0 \Longrightarrow 6x-18+6y=0 \Longrightarrow x=3-y.$$

$$f_y(x,y)=0 \Longrightarrow 6x-6y^2+12y+18=0.$$

Substitution of $x=3-y$ in the second equation gives $y^2-y-6=0$ with solutions $y=3$ or $y=-2$. Therefore we find two critical points: $S_1=(0,3)$ and $S_2=(5,-2)$.

b) For general (x,y) we find $f_{xx}(x,y)=6$, $f_{yy}(x,y)=-12y+12$ and $f_{xy}(x,y)=6=f_{yx}(x,y)$. So we find $f_{xx}(x,y)f_{yy}(x,y)-f_{xy}(x,y)f_{yx}(x,y)=36(1-2y)$. This implies that S_1 is a saddle point ($f_{xx}f_{yy}-f_{xy}f_{yx}<0$) and that f has a local minimum value in S_2 ($f_{xx}f_{yy}-f_{xy}f_{yx}>0$ and $f_{xx}>0$).

c) The unit vector $\mathbf{v}$ in the same direction as $\mathbf{u}$ is given by

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{5}\begin{pmatrix}-3\\4\end{pmatrix}. \text{ Furthermore } \nabla f(1,2) = \begin{pmatrix}f_x(1,2)\\f_y(1,2)\end{pmatrix} = \begin{pmatrix}0\\24\end{pmatrix}.$$

So $D_{\mathbf{v}}(1,2) = \mathbf{v} \bullet \nabla f(1,2) = \frac{96}{5}$.

7. a) Make a sketch of the domain. Then you can easily verify that

$$\int_0^{\sqrt{\pi}} \int_x^{\sqrt{\pi}} y^2 \cos(y^2)\, dy\, dx = \int_0^{\sqrt{\pi}} \int_0^{y} y^2 \cos(y^2)\, dx\, dy$$

$$= \int_0^{\sqrt{\pi}} y^2 \cos(y^2) \Big[x\Big]_{x=0}^{x=y} dy = \int_0^{\sqrt{\pi}} y^3 \cos(y^2)\, dy$$

$$\overset{(*)}{=} \int_0^{\pi} \frac{1}{2} t \cos(t)\, dt \overset{(**)}{=} \frac{1}{2}\Big[t \sin(t) + \cos(t)\Big]_{t=0}^{t=\pi} = -1.$$

At $(*)$ we used the substitution $t = y^2$ and at $(**)$ we used integration by parts.

b) Again sketch the domain. It is that part of the disk around $(0,0)$ with radius $\sqrt{3}$, above the line $y = x$ and right from the y-axis. Using $x = r\cos(\theta), y = r\sin(\theta)$ we get

$$\int\int_S \frac{1}{\sqrt{1+x^2+y^2}}\, dA = \int_0^{\sqrt{3}} \int_{\pi/4}^{\pi/2} \frac{r}{\sqrt{1+r^2}}\, d\theta\, dr =$$

$$= \frac{\pi}{4} \int_0^{\sqrt{3}} \frac{r}{\sqrt{1+r^2}}\, dr = \Big[\frac{\pi}{4}\sqrt{1+r^2}\Big]_0^{\sqrt{3}} = \frac{\pi}{4}(2-1) = \frac{\pi}{4}.$$

8. First calculate $e^{i\pi/3} = \cos(\pi/3) + i\sin(\pi/3) = \frac{1}{2} + \frac{1}{2}i\sqrt{3}$. Use this and multiply numerator and denominator by 2 and multiply the result by the complex conjugate of the denominator. This yields

$$\frac{1}{1-e^{i\pi/3}} = \frac{1}{\frac{1}{2} - \frac{1}{2}i\sqrt{3}} = \frac{2}{1-i\sqrt{3}} = \frac{2}{1-i\sqrt{3}} \cdot \frac{1+i\sqrt{3}}{1+i\sqrt{3}} =$$

$$= \frac{2(1+i\sqrt{3})}{1+3} = \frac{1}{2} + \frac{1}{2}i\sqrt{3}.$$

So $a = \frac{1}{2}$ en $b = \frac{1}{2}\sqrt{3}$.

9. This is a linear differential equation of order one. First divide both sides by x^2 to get

$$y'(x) + \frac{1}{x^2}\, y(x) = \frac{1}{x^2}.$$

Now remark that an integrating factor is $e^{\int \frac{1}{x^2}\, dx} = e^{-\frac{1}{x}}$. This yields

$$\frac{d}{dx}\left(y(x)e^{-\frac{1}{x}}\right) = \frac{1}{x^2}e^{-\frac{1}{x}} \Longrightarrow y(x)e^{-\frac{1}{x}} = \int \frac{1}{x^2}e^{-\frac{1}{x}}\, dx = e^{-\frac{1}{x}} + C.$$

[You can use the substitution $t = -\frac{1}{x}$ if necessary.] So the general solution is

$$y(x) = 1 + Ce^{\frac{1}{x}}, C \in \mathbb{R}.$$

Substitution of the initial value gives

$$2 = y(1) = 1 + Ce, \qquad \text{so} \qquad C = e^{-1}.$$

The solution of the initial value problem is therefore $y(x) = 1 + e^{\frac{1}{x}-1}$.
[This problem can also be solved using separation of the variables!]