Second test Calculus 2, 19 December 2017, Solutions

1. a) Use the chain rule to find

$$f_x(x,y) = \frac{e^{-(x^2/4y)} \cdot \left(\frac{-2x}{4y}\right)}{\sqrt{y}} = \frac{-xe^{-(x^2/4y)}}{2y\sqrt{y}}$$

and therefore (product and chain rule)

$$f_{xx}(x,y) = \frac{-e^{-(x^2/4y)} - xe^{-(x^2/4y)} \cdot \left(\frac{-2x}{4y}\right)}{2y\sqrt{y}} = e^{-(x^2/4y)}\left(\frac{x^2-2y}{4y^2\sqrt{y}}\right).$$

Next use quotient and chain rule to find

$$f_y(x,y) = \frac{e^{-(x^2/4y)} \cdot \left(\frac{x^2}{4y^2}\right)\sqrt{y} - \frac{1}{2\sqrt{y}}e^{-(x^2/4y)}}{y} = e^{-(x^2/4y)}\left(\frac{x^2-2y}{4y^2\sqrt{y}}\right).$$

So we can conclude that
$$\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x^2}.$$

b) The gradient vector at $(2,1)$ is

$$\nabla f(2,1) = f_x(2,1)\,\mathbf{i} + f_y(2,1)\,\mathbf{j} = -e^{-1}\,\mathbf{i} + \frac{1}{2}e^{-1}\,\mathbf{j} = \begin{pmatrix} -e^{-1} \\ \frac{1}{2}e^{-1} \end{pmatrix}.$$

The unit vector $\mathbf{v}$ in the same direction as $\mathbf{u}$ is given by

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

So, since f is clearly differentiable at $(2,1)$, we find that the rate of change of f at $(2,1)$ in the direction of $\mathbf{u}$ is:

$$D_{\mathbf{v}}(2,1) = \mathbf{v} \bullet \nabla f(2,1) = -\frac{e^{-1}}{\sqrt{2}} + \frac{e^{-1}}{2\sqrt{2}} = -\frac{e^{-1}}{2\sqrt{2}} = -\frac{\sqrt{2}}{4e}.$$

2. a) Calculate both first partial derivatives and set them equal to 0:

$$f_x(x,y) = 0 \Longrightarrow 2x - 2y^3 = 0 \Longrightarrow x = y^3.$$

$$f_y(x,y) = 0 \Longrightarrow -6xy^2 + 6y = 0 \Longrightarrow y = 0 \text{ or } y = \frac{1}{x}.$$

Substitution of $y = 0$ in the first equation gives $x = 0$. So we find the critical point $S_1 = (0,0)$. Substitution of $y = \frac{1}{x}$ in the first equation gives $x = \frac{1}{x^3}$ with solution $x = 1$ or $x = -1$. So we also find the critical points $S_2 = (1,1)$ and $S_3 = (-1,-1)$.

b) For general (x, y) we find

$$f_{xx}(x,y) = 2, \;\; f_{yy}(x,y) = 6 - 12xy \quad \text{and} \quad f_{xy}(x,y) = -6y^2 = f_{yx}(x,y).$$

So the determinant of the Hesse matrix is

$$f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)f_{yx}(x,y) = 12 - 24xy - 36y^4.$$

This implies that S_2 and S_3 are saddle points ($f_{xx}f_{yy} - f_{xy}f_{yx} = -48 < 0$) and that f has a local minimum value in S_1 ($f_{xx}f_{yy} - f_{xy}f_{yx} = 12 > 0$ and $f_{xx} = 2 > 0$).

3. Let (x, y, z) be an arbitrary point on the given surface. The distance from this point to the origin is equal to $\sqrt{x^2 + y^2 + z^2}$. We want to minimize this distance subject to the constraint $xy + z^2 = 2$. However, since it will lead to less difficult calculations, we choose to minimize the square of the distance. Therefore introduce the Lagrange function $L(x, y, \lambda) = x^2 + y^2 + z^2 + \lambda(xy + z^2 - 4)$ and find its critical points:

$$\begin{cases} 0 = \frac{\partial L}{\partial x} = 2x + \lambda y & (A) \\ 0 = \frac{\partial L}{\partial y} = 2y + \lambda x & (B) \\ 0 = \frac{\partial L}{\partial z} = 2z + 2\lambda z & (C) \\ 0 = \frac{\partial L}{\partial \lambda} = xy + z^2 - 4 & (D) \end{cases}$$

Equation (C) yields to $\lambda = -1$ (I) or $z = 0$ (II). Consider both cases separately, starting with case (I). Substitution of $\lambda = -1$ in equations (A) and (B) yields $2x = y$ and $2y = x$, so $x = y = 0$, and therefore (use equation (D)) $z = 2$ or $z = -2$. For these points ($(0, 0, -2)$ and $(0, 0, 2)$) the square of the distance to the origin is 4.
Now consider case (II). Substitution of $z = 0$ in equation (D) yields $xy = 4$. Next multiply equation (A) by x and equation (B) by y and subtract the resulting equations to get $2x^2 - 2y^2 = 0$, so $x = y$ or $x = -y$. In combination with $xy = 4$ only $x = y$ is possible and yields $x = y = 2$ or $x = y = -2$. So we also find the critical points $(2, 2, 0)$ and $(-2, -2, 0)$. The square of the distance to the origin for these points is 8.
So in case (I) we found the points that are closest to the origin: $(0, 0, 2)$ and $(0, 0, -2)$.

4. Since we cannot find an antiderivative of $\frac{\sqrt{x}}{1+x^2}$ easily we will reverse the order of integration. Make a sketch of the domain and find:

$$\int_0^2 \int_{y^2}^4 \frac{\sqrt{x}}{1+x^2}\, dx\, dy = \int_0^4 \int_0^{\sqrt{x}} \frac{\sqrt{x}}{1+x^2}\, dy\, dx = \int_0^4 \frac{\sqrt{x}}{1+x^2} \Big[y \Big]_{y=0}^{y=\sqrt{x}} dx$$
$$= \int_0^4 \frac{x}{1+x^2}\, dx = \left[\frac{1}{2} \ln(1+x^2) \right]_{x=0}^{x=4} = \frac{1}{2} \ln(17).$$

5. We will use the transformation $u = \frac{y}{x^2}$ and $v = xy$. Then the region in the (u, v)-plane is given by $1 \le u \le 3$ and $1 \le v \le 3$. Furthermore:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -\frac{2y}{x^3} & \frac{1}{x^2} \\ y & x \end{vmatrix} = -\frac{3y}{x^2} = -3u,$$

which implies that

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} = \left|-\frac{1}{3u}\right| = \frac{1}{3u}.$$

So the integral becomes

$$\iint_D dA = \int_1^3\int_1^3 \frac{1}{3u}\,du\,dv = \int_1^3 \left[\frac{1}{3}\ln(u)\right]_1^3 dv = \int_1^3 \frac{1}{3}\ln 3\,dv = \frac{1}{3}\ln 3\Big[v\Big]_1^3 = \frac{2}{3}\ln 3.$$

6. Rewrite the equation to $z(z^3+i)=0$, which already gives the solution $z_1=0$. For the other (three) solutions we have to solve the equation $z^3=-i$, which is equivalent to solving separately

$$|z^3| = |-i| = 1 \qquad \text{and} \qquad \arg(z^3) = \arg(-i) + 2k\pi = -\frac{\pi}{2} + 2k\pi.$$

Since $|z^3| = |z|^3$, the first equation yields $|z|=1$. And since $\arg(z^3) = 3\arg(z)$ the second equation gives $\arg(z) = -\frac{\pi}{6} + \frac{2}{3}k\pi$. So the other three solutions are (choose $k=0,1,2$):

$$\begin{aligned}
z_2 &= 1\cdot\left(\cos\left(-\frac{1}{6}\pi\right) + i\sin\left(-\frac{1}{6}\pi\right)\right) = \frac{1}{2}\sqrt{3} - \frac{1}{2}i,\\
z_3 &= 1\cdot\left(\cos\left(\frac{1}{2}\pi\right) + i\sin\left(\frac{1}{2}\pi\right)\right) = i,\\
z_4 &= 1\cdot\left(\cos\left(\frac{7}{6}\pi\right) + i\sin\left(\frac{7}{6}\pi\right)\right) = -\frac{1}{2}\sqrt{3} - \frac{1}{2}i.
\end{aligned}$$

7. Rewrite the equation as $r(1+\sin\theta) = 2$, which leads to $r = 2 - r\sin\theta$. The transformation from polar to rectangular coordinates gives $\sqrt{x^2+y^2} = 2-y$. Now square both sides to obtain

$$x^2+y^2 = (2-y)^2 \Longrightarrow x^2 = 4-4y \Longrightarrow y = 1 - \frac{1}{4}x^2.$$

So the resulting curve is a parabola with top $(0,1)$.

8. This is a first order differential equation that is separable. Furthermore it is clear from the initial value that $y(x)=0$ is not a solution. So separate the variables to find:

$$\int \frac{1}{y^2}\,dy = \int x^2\,dx \Longrightarrow \frac{-1}{y} = \frac{1}{3}x^3 + C,$$

so the general solution is $y(x) = \dfrac{-1}{C+\frac{1}{3}x^3}$. Substitute the initial value to obtain $1 = \frac{-1}{C}$, so $C=-1$. So the final solution is $y(x) = \dfrac{-1}{-1+\frac{1}{3}x^3} = \dfrac{3}{3-x^3}$.

9. Substitute $y(x) = e^{rx}$. Then the auxiliary equation becomes

$$4r^2 + 4r + 1 = (2r + 1)^2 = 0,$$

with only one solutions $r = -\frac{1}{2}$. So the general real solution is:

$$y(x) = c_1 e^{x/2} + c_2 x e^{x/2}, c_1, c_2 \in \mathbb{R}.$$