First test Calculus 2, 20 November 2017, Solutions

1. Define the function $f : [2, \infty) \to \mathbb{R}$ by $f(x) = \frac{x^2}{e^x} = x^2 e^{-x}$. Since

$$f'(x) = 2xe^{-x} - x^2 e^{-x} = x(2-x)e^{-x} < 0 \text{ for all } x > 2,$$

f is decreasing on its domain. Therefore the sequence $\left\{\frac{n^2}{e^n}\right\}_{n=2}^{\infty}$ is also decreasing. Furthermore, since $\lim_{n\to\infty} \frac{n^2}{e^n} = 0$ ("e^n wins from n^2", or use $f(x)$ and apply l'Hospitals rule twice), the sequence is convergent. As a consequence the sequence is also bounded (above by its maximum $4e^{-2}$ and below by 0).

2. First notice that $\dfrac{1}{n^2 - 7n + 12} = \dfrac{1}{n-4} - \dfrac{1}{n-3}$. Now look at the partial sums:

$$\sum_{n=5}^{N} \frac{1}{n^2 - 7n + 12} = \sum_{n=5}^{N} \left(\frac{1}{n-4} - \frac{1}{n-3}\right) =$$

$$= \left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \ldots + \frac{1}{N-5} - \frac{1}{N-4} + \frac{1}{N-4} - \frac{1}{N-3}\right) = 1 - \frac{1}{N-3}.$$

So

$$\sum_{n=5}^{\infty} \frac{1}{n^2 - 7n + 12} = \lim_{n\to\infty} \sum_{n=5}^{N} \left(\frac{1}{n-4} - \frac{1}{n-3}\right) = \lim_{n\to\infty} \left(1 - \frac{1}{N-3}\right) = 1.$$

3. a) Let $a_n = \dfrac{n + \sqrt{n}}{2n^2 - n + 3}$ and choose $b_n = \dfrac{1}{n}$. Then use the limit comparison test:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n + \sqrt{n}}{2n^2 - n + 3} \div \frac{1}{n} = \lim_{n\to\infty} \frac{n^2 + n\sqrt{n}}{2n^2 - n + 3} = \lim_{n\to\infty} \frac{1 + \frac{1}{\sqrt{n}}}{2 - \frac{1}{n} + \frac{3}{n^2}} = \frac{1}{2}.$$

Since $\sum_{n=1}^{\infty} b_n$ is divergent (p-series with $p = 1$), $\sum_{n=1}^{\infty} \dfrac{n + \sqrt{n}}{2n^2 - n + 3}$ is also divergent.

b) Use the ratio test and the fact that $(n+1)! = (n+1)n!$ to find that

$$\lim_{n\to\infty} \frac{2^{(n+1)^2}}{(n+1)!} \div \frac{2^{n^2}}{n!} = \lim_{n\to\infty} \frac{2^{2n+1}}{n+1} = \infty > 1,$$

since "2^n wins from n" (or use l'Hospitals rule). So the series is divergent.

4. We use the ratio test:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{2^{n+1}(4-x)^{n+1}}{\sqrt{n+1}} \div \frac{2^n (4-x)^n}{\sqrt{n}} \right| \\ &= |2(4-x)| \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = |2(4-x)|. \end{aligned}$$

So the series converges absolutely for $|2(4-x)| < 1$, that is for $\frac{7}{2} < x < \frac{9}{2}$, and diverges for $|2(4-x)| > 1$, which is for $x < \frac{7}{2}$ or for $x > \frac{9}{2}$.

Now determine separately the behavior in the endpoints: First substitute $x = \frac{7}{2}$ to find $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is a divergent series (p-series with $p = \frac{1}{2}$). Then substitute $x = \frac{9}{2}$ to get $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. With the alternating series test (the series is clearly alternating and the sequence $\{\frac{1}{\sqrt{n}}\}$ is decreasing with limit 0) we can conclude that this series converges (conditionally). So the interval of convergence is $(\frac{7}{2}, \frac{9}{2}]$.

5. a) We start with the (well-known) Maclaurin series

$$\cos t = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n},$$

which converges for all $t \in \mathbb{R}$ to $\cos t$, and use the substitution $t = 2x$ to get

$$x^3 \cos(2x) = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (2x)^{2n} = \sum_{n=0}^{\infty} \frac{(-4)^n x^{2n+3}}{(2n)!},$$

exactly as required (we used $2^{2n} = 4^n$).

b) Take derivatives on both sides of the identity

$$x^3 \cos(2x) = \sum_{n=0}^{\infty} \frac{(-4)^n x^{2n+3}}{(2n)!}$$

to find (use term-by-term differentiation in the series)

$$3x^2 \cos(2x) - 2x^3 \sin(2x) = \sum_{n=0}^{\infty} \frac{(-4)^n (2n+3)}{(2n)!} x^{2n+2}.$$

Now divide by x^2 on both sides to get:

$$3\cos(2x) - 2x \sin(2x) = \sum_{n=0}^{\infty} \frac{(-4)^n (2n+3)}{(2n)!} x^{2n}$$

and finally substitute $x = \frac{\pi}{2}$:

$$3\cos(\pi) - \pi \sin(\pi) = \sum_{n=0}^{\infty} \frac{(-4)^n (2n+3)}{(2n)!} \left(\frac{\pi}{2}\right)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+3)\pi^{2n}}{(2n)!},$$

so that

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2n+3)\pi^{2n}}{(2n)!} = -3.$$

6. a) $\mathbf{u} \bullet \mathbf{v} = 0 + 0 - 2 = -2$ and $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & -1 \\ 2 & 0 & 2 \end{vmatrix} = 6\mathbf{i} - 2\mathbf{j} - 6\mathbf{k} = \begin{pmatrix} 6 \\ -2 \\ -6 \end{pmatrix}$.

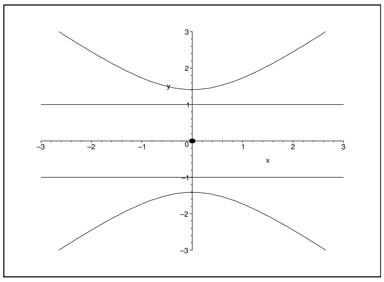
b) $\mathbf{u_v} = \dfrac{\mathbf{u} \bullet \mathbf{v}}{\mathbf{v} \bullet \mathbf{v}} \mathbf{v} = -\dfrac{2}{8}\mathbf{v} = \begin{pmatrix} -1/2 \\ 0 \\ -1/2 \end{pmatrix} = -\dfrac{1}{2}\mathbf{i} - \dfrac{1}{2}\mathbf{k}.$

c) $\mathbf{u} \bullet \begin{pmatrix} x+1 \\ y-3 \\ z-1 \end{pmatrix} = 0$, so $0 \cdot (x+1) + 3 \cdot (y-3) - 1 \cdot (z-1) = 0$, or $3y - z = 8$.

d) The distance is given by

$$\frac{|(0 \cdot 1) + (3 \cdot 2) + (-1 \cdot 3) - 8|}{\sqrt{0^2 + 3^2 + (-1)^2}} = \frac{5}{\sqrt{10}} = \frac{1}{2}\sqrt{10}.$$

7. a) For $c = 0$ the level "curve" is the point $(0, 0)$. For $c = 1$ we find the lines $y = 1$ and $y = -1$. For $c = 2$ we find the hyperbola $y^2 - x^2 = 2$.



b) The partial derivatives are $\dfrac{\partial f}{\partial x} = \dfrac{2x(1-y^2)}{(x^2+1)^2}$ and $\dfrac{\partial f}{\partial y} = \dfrac{2y}{x^2+1}$.

c) The tangent plane passes through $P = (1, 0, \frac{1}{2})$. Furthermore: $\frac{\partial f}{\partial x}(1,0) = \frac{1}{2}$ and $\frac{\partial f}{\partial y}(1,0) = 0$. So an equation of the tangent plane is

$$z = \frac{1}{2} + \frac{1}{2}(x-1) + 0(y-0) = \frac{1}{2}x, \text{ or } x - 2z = 0.$$