Resit Calculus 2, 16 February 2017, Solutions

1. a) Let $a_n = \frac{e^{n^2}}{n!}$ and remark that $(n+1)! = (n+1)n!$. Now use the ratio-test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{e^{(n+1)^2}}{(n+1)!} \div \frac{e^{n^2}}{n!} = \lim_{n\to\infty} \frac{e^{2n+1}}{n+1} = \infty > 1.$$

Therefore the series is divergent.

b) First consider $\sum_{n=1}^{\infty} \frac{\ln n}{n}$. Since $\frac{\ln n}{n} > \frac{1}{n}$ and since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (harmonic series), the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is also divergent (comparison test). So the series $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$ is not absolutely convergent. Now use the alternating series test: (1) the series is alternating, (2) the sequence $\left\{\frac{\ln n}{n}\right\}$ is ultimately decreasing (prove this for example by looking at $f(x) = \frac{\ln x}{x}$, with $f'(x) = \frac{1-\ln x}{x^2} < 0$ for $x > e$), and (3) the sequence $\left\{\frac{\ln n}{n}\right\}$ has limit 0. So the series $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$ is convergent. Finally we can conclude that the series is conditionally convergent.

2. a) We use the ratio test:

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(x-1)^{2(n+1)}}{(n+1)4^{n+1}}\right| \div \left|\frac{(x-1)^{2n}}{n4^n}\right| \\ &= \frac{|x-1|^2}{4} \lim_{n\to\infty} \frac{n}{n+1} = \frac{|x-1|^2}{4}.\end{aligned}$$

So the series converges absolutely for $|x-1|^2 < 4$, that is for $-1 < x < 3$, and diverges for $|x-1|^2 > 4$. Now determine the behavior in the endpoints: For both $x = -1$ and $x = 3$ we find the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ which is divergent. So the interval of convergence is $(-1, 3)$.

b) Inside the interval of convergence we can use term-by-term differentiation:

$$f'(x) = \sum_{n=1}^{\infty} \frac{2n(x-1)^{2n-1}}{n4^n} = 2\sum_{n=1}^{\infty} \frac{(x-1)^{2n-1}}{4^n}.$$

Therefore (substitute $x = 0$ and use $(-1)^{2n-1} = -1$):

$$f'(0) = 2\sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{4^n} = -2\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = -2 \cdot \frac{1}{4} \cdot \frac{1}{1-\frac{1}{4}} = -\frac{2}{3}.$$

3. First rewrite $f(x)$ to $\frac{x^2}{2} \cdot \frac{1}{1-\frac{x}{2}}$. Now use the geometric series $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$, which converges for all $t \in (-1, 1)$. This yields (substitute $t = \frac{x}{2}$)

$$f(x) = \frac{x^2}{2-x} = \frac{x^2}{2} \cdot \frac{1}{1-\frac{x}{2}} = \frac{x^2}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^{n+2}}{2^{n+1}},$$

converging for $\frac{x}{2} \in (-1, 1)$, so for $x \in (-2, 2)$.

4. a) $\mathbf{u} \bullet \mathbf{v} = -2 + 0 + 0 = -2$ and $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix} = 3\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$.

b) All vectors that are perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ are multiples of the cross-product $\mathbf{u} \times \mathbf{v}$. So all unit vectors (meaning with length 1) perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ are

$$\pm \frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \pm \frac{1}{\sqrt{22}} \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix} = \pm \left(\frac{3}{\sqrt{22}}\mathbf{i} + \frac{3}{\sqrt{22}}\mathbf{j} - \frac{2}{\sqrt{22}}\mathbf{k} \right).$$

5. Use the chain rule:

$$\frac{\partial}{\partial y} f(x^2y, xy^2) = x^2 f_1(x^2y, xy^2) + 2xy f_2(x^2y, xy^2).$$

6. a) Calculate both first partial derivatives and set them equal to 0:

$$f_x(x,y) = 0 \Longrightarrow 3x^2 - 6y = 0 \Longrightarrow y = \frac{1}{2}x^2.$$

$$f_y(x,y) = 0 \Longrightarrow -6x + 3y^2 = 0 \Longrightarrow x = \frac{1}{2}y^2.$$

Substitution of $y = \frac{1}{2}x^2$ in the second equation gives $x = \frac{1}{8}x^4$ with solutions $x = 0$ or $x = 2$. Therefore we find two critical points: $S_1 = (0,0)$ and $S_2 = (2,2)$.

b) For general (x,y) we find $f_{xx}(x,y) = 6x$, $f_{yy}(x,y) = 6y$ and $f_{xy}(x,y) = -6 = f_{yx}(x,y)$. So we find $f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)f_{yx}(x,y) = 36(xy-1)$. This implies that S_1 is a saddle point ($f_{xx}f_{yy} - f_{xy}f_{yx} < 0$) and that f has a local minimum value in S_2 ($f_{xx}f_{yy} - f_{xy}f_{yx} > 0$ and $f_{xx} > 0$).

c) The unit vector $\mathbf{v}$ in the same direction as $\mathbf{u}$ is given by

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \begin{pmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix}. \text{ Furthermore } \nabla f(1,1) = \begin{pmatrix} -3 \\ -3 \end{pmatrix}.$$

So $D_{\mathbf{v}}(1,1) = \mathbf{v} \bullet \nabla f(1,1) = \frac{-12}{\sqrt{10}}$.

7. a) Make a sketch of the domain. Then you can easily verify that

$$\int_0^2 \int_y^2 \frac{x^2}{1+x^4}\,dx\,dy = \int_0^2 \int_0^x \frac{x^2}{1+x^4}\,dy\,dx = \int_0^2 \frac{x^2}{1+x^4}\Big[y\Big]_{y=0}^{y=x} dx$$
$$= \int_0^2 \frac{x^3}{1+x^4}\,dx = \left[\frac{1}{4}\ln(1+x^4)\right]_{x=0}^{x=2} = \frac{1}{4}\ln 17.$$

b) Again sketch the domain. It is the region between the circles around $(0,0)$ with radius 2 respectively 3, above the x-axis and left from the y-axis. Using $x = r\cos(\theta), y = r\sin(\theta)$ and integration by parts we get

$$\int\int_S \cos\left(\pi\sqrt{x^2+y^2}\right) dA = \int_2^3 \int_{\pi/2}^{\pi} r\cos(\pi r)\,d\theta\,dr = \int_2^3 r\cos(\pi r)\Big[\theta\Big]_{\theta=\pi/2}^{\theta=\pi} dr$$
$$= \frac{\pi}{2}\left(\left[\frac{r}{\pi}\sin(\pi r)\right]_2^3 - \frac{1}{\pi}\int_2^3 \sin(\pi r)\,dr\right) = \frac{\pi}{2}\frac{1}{\pi^2}\Big[\cos(\pi r)\Big]_2^3 = -\frac{1}{\pi}$$

8. Suppose $z = 1 - i\sqrt{3}$ and $w = \sqrt{3} + i$. Then $|z| = \sqrt{1+3} = 2$, $|w| = \sqrt{3+1} = 2$ and $\arg(z) = -\frac{1}{3}\pi$, $\arg(w) = \frac{1}{6}\pi$. This implies that

$$\left|\frac{z^9}{w^6}\right| = \frac{|z|^9}{|w|^6} = \frac{2^9}{2^6} = 8.$$

And for the argument we have

$$\arg\left(\frac{z^9}{w^6}\right) = 9\arg(z) - 6\arg(w) = -\frac{9}{3}\pi - \frac{6}{6}\pi = -4\pi,$$

which is equivalent to 0 radians. So

$$\frac{z^9}{w^6} = 8(\cos 0 + i\sin 0) = 8.$$

So $a = 8$ and $b = 0$.

9. This is a linear differential equation of order one. First divide both sides by x to get

$$\frac{dy}{dx} - \frac{1}{x}y(x) = xe^{2x}.$$

Since $\int -\frac{1}{x}\,dx = -\ln x = \ln\left(\frac{1}{x}\right)$, the integrating factor is $e^{\ln\left(\frac{1}{x}\right)} = \frac{1}{x}$. So we can rewrite this equation into

$$\left(\frac{1}{x}y(x)\right)' = e^{2x},$$

with general solution

$$\frac{1}{x}y(x) = \frac{1}{2}e^{2x} + C, \text{ thus } y(x) = \frac{1}{2}xe^{2x} + Cx,\ C \in \mathbb{R}.$$

Substitution of the initial value gives $3e^4 = y(2) = e^4 + 2C$, so $C = e^4$. The solution is therefore:

$$y(x) = \frac{1}{2}xe^{2x} + xe^4.$$