Second test Calculus 2, 22 December 2016, Solutions

1. a) Since $f_x(x,y) = y^3\cos(xy)$ and $f_y(x,y) = 2y\sin(xy) + xy^2\cos(xy)$ the gradient at $(\pi, 1)$ is

$$\nabla f(\pi,1) = f_x(\pi,1)\mathbf{i} + f_y(\pi,1)\mathbf{j} = -\mathbf{i} - \pi\mathbf{j} = \begin{pmatrix} -1 \\ -\pi \end{pmatrix}.$$

b) The unit vector $\mathbf{v}$ in the same direction as $\mathbf{u}$ is given by

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}.$$

So, since f is clearly differentiable at $(\pi, 1)$, we find

$$D_{\mathbf{v}}(\pi,1) = \mathbf{v} \bullet \nabla f(\pi,1) = -\frac{2}{\sqrt{5}} - \frac{\pi}{\sqrt{5}} = -\frac{2+\pi}{\sqrt{5}}.$$

2. a) Calculate both first partial derivatives and set them equal to 0:

$$f_x(x,y) = 0 \Longrightarrow 3x^2 - 3y - 9 = 0 \Longrightarrow y = x^2 - 3.$$

$$f_y(x,y) = 0 \Longrightarrow -3x + 6y = 0 \Longleftrightarrow x = 2y.$$

Substitution of $x = 2y$ in the first equation gives $4y^2 - y - 3 = (4y+3)(y-1) = 0$. So we find two critical points: $S_1 = (2,1)$ and $S_2 = (-\frac{3}{2}, -\frac{3}{4})$.

b) For general (x, y) we find

$$f_{xx}(x,y) = 6x, f_{yy}(x,y) = 6 \quad \text{and} \quad f_{xy}(x,y) = -3 = f_{yx}(x,y).$$

So the determinant of the Hesse matrix is

$$f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)f_{yx}(x,y) = 36x - 9.$$

This implies that S_2 is a saddle point ($f_{xx}f_{yy} - f_{xy}f_{yx} = -63 < 0$) and that f has a local minimum value in S_1 ($f_{xx}f_{yy} - f_{xy}f_{yx} = 63 > 0$ and $f_{xx} = 12 > 0$).

3. Introduce a function $L(x,y,\lambda) = 5x - 3y + \lambda(x^2 + y^2 - 34)$ and find its critical points:

$$\begin{cases} 0 = \frac{\partial L}{\partial x} = 5 + 2\lambda x & (A) \\ 0 = \frac{\partial L}{\partial y} = -3 + 2\lambda y & (B) \\ 0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 - 34 & (C) \end{cases}$$

Since clearly $\lambda \neq 0$ equations (A) and (B) can be rewritten as

$$x = -\frac{5}{2\lambda} \quad \text{and} \quad y = \frac{3}{2\lambda}.$$

Substitution of these values in equation (C) gives

$$\frac{25}{4\lambda^2} + \frac{9}{4\lambda^2} = 34, \quad \text{so} \quad \lambda = \pm\frac{1}{2},$$

which results in two critical points: $S_1 = (x,y,\lambda) = (5, -3, -\frac{1}{2})$ and $S_2 = (x,y,\lambda) = (-5, 3, \frac{1}{2})$. In S_1 the function has its maximum value $f(5,-3) = 34$ and in S_2 the function has its minimum value $f(-5,3) = -34$.

4. a) Make a sketch of the domain. Since we cannot antidifferentiate $\sqrt{x}e^x$ with respect to x the inner integral should be over y. We find (use partial integration in the second integral):

$$\int\int_R \sqrt{x}\, e^x\, dA = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} \sqrt{x}e^x\, dy\, dx = \int_0^4 \sqrt{x}e^x \Big[y\Big]_{y=-\sqrt{x}}^{y=\sqrt{x}} dx$$
$$= \int_0^4 2xe^x\, dx = \Big[2xe^x - 2e^x\Big]_{x=0}^{x=4} = 6e^4 + 2.$$

b) Again sketch the domain. S is the part of the disc around $(0,0)$ with radius 2, in the first quadrant and above the line $y = x$. Using $x = r\cos(\theta), y = r\sin(\theta)$ we get

$$\int\int_S \frac{1}{1+x^2+y^2}\, dA = \int_{\pi/4}^{\pi/2} \int_0^2 \frac{r}{1+r^2}\, dr\, d\theta =$$
$$= \int_{\pi/4}^{\pi/2} \left[\frac{1}{2}\ln(1+r^2)\right]_{r=0}^{r=2} = \frac{1}{2}\ln 5 \Big[\theta\Big]_{\theta=\pi/4}^{\theta=\pi/2} = \frac{\pi}{8}\ln 5.$$

5. a) $|w| = \sqrt{1+3} = 2$ and $\arg(w) = -\frac{1}{3}\pi$.

b) Assume $z = re^{i\phi}$, with $r > 0$. Then the equation is

$$z^3 = r^3 e^{3i\phi} = 8i = 8e^{i\pi/2} = 8e^{i\pi/2+2ki\pi}, \qquad \text{with } k \in \mathbb{Z}.$$

This implies that $r = 2$ and $\phi = \frac{\pi}{6} + \frac{2k\pi}{3}$, $k \in \mathbb{Z}$. So the solutions are (choose $k = 0, 1, 2$):

$$z_1 = 2e^{i\pi/6} = \sqrt{3} + i,\ z_2 = 2e^{5i\pi/6} = -\sqrt{3} + i \text{ and } z_3 = 2e^{3i\pi/2} = -2i.$$

[An equivalent method is solving $|z^3| = |8i|$ and $\arg(z^3) = \arg(8i) + 2k\pi$ separately.]

6. First multiply the left hand side and right hand side of the equation with r. Then substitute $x = r\cos\theta$ en $y = r\sin\theta$ (so we have $x^2 + y^2 = r^2$). This gives:

$$r^2 = 2r\cos\theta \iff x^2 + y^2 = 2x \iff (x-1)^2 + y^2 = 1.$$

So the resulting curve is a circle with center $(1, 0)$ and radius 1.

7. This is a first order linear differential equation which can be solved by using an integrating factor. First divide the equation by $\sqrt{x}$ and conclude that the integrating factor is $e^{-2\sqrt{x}}$. This gives

$$\frac{d}{dx}\left(y(x)e^{-2\sqrt{x}}\right) = \frac{e^{-\sqrt{x}}}{\sqrt{x}} \Longrightarrow y(x)e^{-2\sqrt{x}} = \int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx = -2e^{-\sqrt{x}} + C.$$

[The last integral can be calculated using the substitution $t = \sqrt{x}$.] So the general solution becomes

$$y(x) = -2e^{\sqrt{x}} + Ce^{2\sqrt{x}}, C \in \mathbb{R}.$$

8. Substitute $y(x) = e^{rx}$. Then the auxiliary equation becomes

$$r^2 - 4r + 13 = (r-2)^2 + 9 = 0,$$

with complex solutions $r = 2 \pm 3i$. So the general (real) solution is:

$$y(x) = c_1 e^{2x} \cos(3x) + c_2 e^{2x} \sin(3x), c_1, c_2 \in \mathbb{R}.$$

Substitution of the first initial value gives $0 = y(0) = c_1$. Now

$$y'(x) = 2c_2 e^{2x} \sin(3x) + 3c_2 e^{2x} \cos(3x),$$

so substitution of the second initial value gives $6 = y'(0) = 3c_2$, so $c_2 = 2$. The solution of the initial value problem is therefore $y(x) = 2e^{2x} \sin(3x)$.