First test Calculus 2, 21 November 2016, Solutions

1. Use the fact that $2^{2n} = 4^n$ and then calculate the limit by dividing numerator and denominator by 4^n. We find:

$$\lim_{n\to\infty} \frac{3^n + 4^n + n}{1 + 2^{2n} + n^3} = \lim_{n\to\infty} \frac{\left(\frac{3}{4}\right)^n + 1 + \frac{n}{4^n}}{\left(\frac{1}{4}\right)^n + 1 + \frac{n^3}{4^n}} = 1$$

since $\lim_{n\to\infty} x^n = 0$ for all $|x| < 1$ and $\lim_{n\to\infty} \frac{n^k}{4^n} = 0$ for all k (both standard limits). So the sequence converges with limit 1.

2. Note that this is a geometrical series. So after rewriting and shifting the index we have (use $4^{2n} = 16^n$)

$$\sum_{n=1}^{\infty} 3^{n-1} 4^{1-2n} = \sum_{n=1}^{\infty} 3^{-1} 4^1 \left(\frac{3}{16}\right)^n = \frac{4}{3} \sum_{n=0}^{\infty} \left(\frac{3}{16}\right)^{n+1} = \frac{4}{3} \cdot \frac{3}{16} \cdot \frac{1}{1 - \frac{3}{16}} = \frac{4}{13}.$$

3. a) Let $a_n = \dfrac{n}{n^2 + n - 1}$ and choose $b_n = \dfrac{1}{n}$. Then use the limit comparison test:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^2 + n - 1} \div \frac{1}{n} = \lim_{n\to\infty} \frac{n^2}{n^2 + n - 1} = \lim_{n\to\infty} \frac{1}{1 + \frac{1}{n} - \frac{1}{n^2}} = 1.$$

Since $\sum_{n=1}^{\infty} b_n$ is divergent (p-series with $p = 1$), $\sum_{n=1}^{\infty} \dfrac{n}{n^2 + n - 1}$ is also divergent.

b) Since $\lim_{n\to\infty} \cos\left(\frac{1}{n^2}\right) = \cos(0) = 1 \neq 0$ the series is divergent (nth-term test).

c) Use the ratio test and the fact that $(n+1)! = (n+1)n!$ and $(2n+2)! = (2n+2)(2n+1)(2n)!$ to find that

$$\lim_{n\to\infty} \frac{(2n+2)!}{(n+1)!(n+1)!} \div \frac{(2n)!}{n!n!} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \lim_{n\to\infty} 2 \cdot \frac{2 + \frac{1}{n}}{1 + \frac{1}{n}} = 4 > 1,$$

so the series is divergent.

4. We use the ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(3x-1)^{n+1}}{2^{n+1} \ln(n+1)} \div \frac{(3x-1)^n}{2^n \ln(n)} \right|$$
$$= \left| \frac{3x-1}{2} \right| \lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)} = \left| \frac{3x-1}{2} \right|.$$

[You can use l'Hospital for calculating the last limit.] So the series converges absolutely for $|3x - 1| < 2$, that is for $-\frac{1}{3} < x < 1$, and diverges for $|3x - 1| > 2$. Now determine separately the behavior in the endpoints: First take $x = 1$. We find $\sum_{n=2}^{\infty} \dfrac{1}{\ln(n)}$ which is a divergent series, since $\frac{1}{\ln(n)} > \frac{1}{n}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges (p-series with $p = 1$). Here we used the comparison test. Then consider $x = -\frac{1}{3}$. We get

$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$. With the alternating series test (the series is clearly alternating and the sequence $\left\{\frac{1}{\ln(n)}\right\}$ is decreasing with limit 0) we can conclude that this series converges (conditionally). So the interval of convergence is $[-\frac{1}{3}, 1)$.

5. a) We need the geometrical series $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$, which converges for all t with $|t| < 1$, but with the substitution $t = \frac{x^2}{2}$ to get $\frac{1}{1-(x^2/2)} = \sum_{n=0}^{\infty} \left(\frac{x^2}{2}\right)^n$, which converges for all x with $|x^2/2| < 1$, so for $|x| < \sqrt{2}$. Then we find

$$f(x) = \frac{x}{2} \cdot \frac{1}{1-(x^2/2)} = \frac{x}{2} \sum_{n=0}^{\infty} \left(\frac{x^2}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^{n+1}} = \sum_{n=0}^{\infty} a_n x^n,$$

converging to $f(x)$ for all x with $|x| < \sqrt{2}$. Now distinguish between even and odd n, to get: $a_{2n} = 0$ for all $n \geq 0$ and $a_{2n+1} = \frac{1}{2^{n+1}}$ for all $n \geq 0$.

b) Since $\sum_{n=0}^{\infty} a_n x^n$ is the Taylor series of $f(x)$ about 0, we know that $a_n = \frac{f^{(n)}(0)}{n!}$. So we need $n = 7$ to find $f^{(7)}(0) = 7! \cdot a_7 = 7! \cdot \frac{1}{2^4} = 315$.

6. a) $\mathbf{u} \bullet \mathbf{v} = -2 + 0 + 0 = -2$ and $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 1 \\ -1 & 3 & 0 \end{vmatrix} = -3\mathbf{i} - \mathbf{j} + 6\mathbf{k} = \begin{pmatrix} -3 \\ -1 \\ 6 \end{pmatrix}$.

b) $\mathbf{u} \bullet \begin{pmatrix} x-1 \\ y-2 \\ z-3 \end{pmatrix} = 0$, so $2 \cdot (x-1) + 0 \cdot (y-2) + 1 \cdot (z-3) = 0$, or $2x + z = 5$.

c) The distance is given by

$$\frac{|(2 \cdot 0) + (0 \cdot 1) + (1 \cdot 0) - 5|}{\sqrt{2^2 + 0^2 + 1^2}} = \frac{5}{\sqrt{5}} = \sqrt{5}.$$

7. a) The partial derivatives are $\frac{\partial f}{\partial x} = \frac{y^2 - x^2 + 1}{(x^2 + y^2 + 1)^2}$ and $\frac{\partial f}{\partial y} = \frac{-2xy}{(x^2 + y^2 + 1)^2}$.

b) The tangent plane passes through $P = (1, 1, \frac{1}{3})$. Furthermore: $\frac{\partial f}{\partial x}(1,1) = \frac{1}{9}$ and $\frac{\partial f}{\partial y}(1,1) = \frac{-2}{9}$. So an equation of the tangent plane is

$$z = \frac{1}{3} + \frac{1}{9}(x-1) - \frac{2}{9}(y-1) = \frac{1}{9}x - \frac{2}{9}y + \frac{4}{9}, \text{ or } x - 2y - 9z + 4 = 0.$$