Resit Calculus 2, 11 February 2016, Solutions

1. a) Let $a_n = \dfrac{1+2n}{n^2 - n\sqrt{n}+3}$, choose $b_n = \dfrac{1}{n}$ and then use the limit comparison test:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+2n^2}{n^2-n\sqrt{n}+3} = \lim_{n\to\infty} \frac{\frac{1}{n}+2}{1-\frac{1}{\sqrt{n}}+\frac{3}{n^2}} = 2.$$

Since $\sum_{n=1}^{\infty} b_n$ is divergent (p-series with $p=1$), the series $\sum_{n=1}^{\infty} \dfrac{1+2n}{n^2-n\sqrt{n}+3}$ is also divergent.

b) Let $a_n = \dfrac{n!}{(2n)!}$ and use $(n+1)! = (n+1)n!$ and $(2(n+1))! = (2n+2)! = (2n+2)(2n+1)(2n)!$. Now use the ratio-test:

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{(n+1)!}{(2(n+1))!} \div \frac{n!}{(2n)!} = \\ &= \lim_{n\to\infty} \frac{n+1}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{1}{2(2n+1)} = 0 < 1.\end{aligned}$$

Therefore the series is (absolute) convergent.

2. We use the ratio test:

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(2x-3)^{n+1}}{4^{n+1}\sqrt{n+1}}\right| \div \left|\frac{(2x-3)^n}{4^n\sqrt{n}}\right| \\ &= \left|\frac{2x-3}{4}\right| \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = \left|\frac{2x-3}{4}\right|.\end{aligned}$$

So the series converges absolutely for $|2x-3| < 4$, that is for $-\frac{1}{2} < x < \frac{7}{2}$, and diverges for $|x-3| > 4$. Now determine separately the behavior in the endpoints: First take $x = \frac{7}{2}$. We find $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$ which is a divergent series (p-series with $p = 1/2$). Then consider $x = -\frac{1}{2}$. We get $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt{n}}$. With the alternating series test we can conclude that this series converges (conditionally). So the interval of convergence is $[-\frac{1}{2}, \frac{7}{2})$.

3. Use the geometric series $\dfrac{1}{1-t} = \sum_{n=0}^{\infty} t^n$, which converges for all $t \in (-1,1)$. Substitute $t = 2x$ and multiply the result by x. This yields

$$f(x) = \frac{x}{1-2x} = x\sum_{n=0}^{\infty}(2x)^n = \sum_{n=0}^{\infty} 2^n x^{n+1} = \sum_{n=1}^{\infty} 2^{n-1}x^n,$$

converging for $2x \in (-1,1)$, so for $x \in (-\frac{1}{2}, \frac{1}{2})$.

4. a) $\mathbf{u} \bullet \mathbf{v} = 3 + 0 - 2 = 1$ and $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} = \mathbf{i} + 5\mathbf{j} + \mathbf{k}$.

b) $\mathbf{u}_{\mathbf{v}} = \dfrac{\mathbf{u} \bullet \mathbf{v}}{\mathbf{v} \bullet \mathbf{v}} \mathbf{v} = \dfrac{1}{2}\mathbf{v} = \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix} = \dfrac{1}{2}\mathbf{i} - \dfrac{1}{2}\mathbf{k}$.

5. a) Calculate both first partial derivatives and set them equal to 0:

$$f_x(x,y) = 0 \Longrightarrow 2x - 2xy = 0 \Longrightarrow x = 0 \text{ or } y = 1.$$

$$f_y(x,y) = 0 \Longrightarrow -x^2 - 2y + 6 = 0.$$

Substitution of $x = 0$ or $y = 1$ in the second equation gives three critical points: $S_1 = (0,3)$, $S_2 = (2,1)$ and $S_3 = (-2,1)$.

b) For general (x,y) we find $f_{xx}(x,y) = 2 - 2y, f_{yy}(x,y) = -2$ and $f_{xy}(x,y) = -2x = f_{yx}(x,y)$. So we find $f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)f_{yx}(x,y) = 4(y - x^2 - 1)$. This implies that S_2 and S_3 are saddle points ($f_{xx}f_{yy} - f_{xy}f_{yx} < 0$) and that f has a local maximum value in S_1 ($f_{xx}f_{yy} - f_{xy}f_{yx} > 0$ and $f_{xx} < 0$).

c) The equation of the tangent plane is:

$$z = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) = 3 + 0(x-1) + 3(y-1) = 3y.$$

6. a) Make a sketch of the domain. Then you can easily verify that

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\, dx\, dy = \int_0^1 \int_0^{x^2} \sqrt{x^3+1}\, dy\, dx = \int_0^1 \sqrt{x^3+1}\Big[y\Big]_{y=0}^{y=x^2} dx$$

$$= \int_0^1 x^2\sqrt{x^3+1}\, dx = \left[\frac{2}{9}\left(x^3+1\right)^{3/2}\right]_{x=0}^{x=1} = \frac{2}{9}\left(2\sqrt{2}-1\right).$$

b) Again sketch the domain. It is the part of the disc around $(0,0)$ with radius 2, under the line $y = x$ and above the line $y = -x$. Using $x = r\cos(\theta), y = r\sin(\theta)$ we get

$$\int\int_S \frac{1}{\sqrt{1+x^2+y^2}}\, dA = \int_{-\pi/4}^{\pi/4} \int_0^2 \frac{r}{\sqrt{1+r^2}}\, dr\, d\theta = \int_{-\pi/4}^{\pi/4} \left[\sqrt{1+r^2}\right]_{r=0}^{r=2}$$

$$= \int_{-\pi/4}^{\pi/4} (\sqrt{5}-1)\, d\theta = (\sqrt{5}-1)[\theta\Big]_{\theta=-\pi/4}^{\theta=\pi/4} = \frac{1}{2}\pi(\sqrt{5}-1).$$

7. Assume $z = re^{i\phi}$, with $r > 0$. Then our equation is $z^3 = r^3e^{3i\phi} = -8i = 8e^{-i\pi/2} = 8e^{-i\pi/2+2ki\pi}$ with $k \in \mathbb{Z}$. This implies that $r = 2$ and $\phi = -\frac{\pi}{6} + \frac{2k\pi}{3}$, $k \in \mathbb{Z}$. So the solutions are (choose $k = 0,1,2$):

$$z_1 = 2e^{-i\pi/6} = \sqrt{3} - i,\ z_2 = 2e^{i\pi/2} = 2i \text{ and } z_3 = 2e^{7i\pi/6} = -\sqrt{3} - i.$$

8. This is a linear differential equation of order one. Since $\int -2x\,dx = -x^2$, the integrating factor is e^{-x^2}. So we can rewrite this equation into

$$\left(e^{-x^2} y(x)\right)' = \cos(2x),$$

with general solution

$$e^{-x^2} y(x) = \frac{1}{2}\sin(2x) + C, \text{ thus } y(x) = \frac{1}{2}e^{x^2}\sin(2x) + Ce^{x^2},\ C \in \mathbb{R}.$$

Substitution of the initial value gives $1 = y(0) = 0 + C$, so $C = 1$. The solution is therefore:

$$y(x) = e^{x^2}\left(\frac{1}{2}\sin(2x) + 1\right).$$