Second test Calculus 2, 17 December 2015, Solutions

1. First
$$\frac{\partial}{\partial x} f(xy^2, x^2y) = y^2 f_1(xy^2, x^2y) + 2xy f_2(xy^2, x^2y).$$
Then
$$\frac{\partial^2}{\partial y \partial x} f(xy^2, x^2y) = \frac{\partial}{\partial y}\left(y^2 f_1(xy^2, x^2y) + 2xy f_2(xy^2, x^2y)\right)$$
$$= 2y f_1(xy^2, x^2y) + y^2\left(2xy f_{11}(xy^2, x^2y) + x^2 f_{12}(xy^2, x^2y)\right)$$
$$+2x f_2(xy^2, x^2y) + 2xy\left(2xy f_{21}(xy^2, x^2y) + x^2\ f_{22}(xy^2, x^2y)\right).$$

2. a) Calculate both first partial derivatives and set them equal to 0:
$$f_x(x, y) = 0 \Longrightarrow 2xy - 2x = 0 \Longrightarrow x = 0 \text{ or } y = 1.$$
$$f_y(x, y) = 0 \Longrightarrow x^2 + 2y - 4 = 0.$$
Substitution of $x = 0$ or $y = 1$ in the second equation gives three critical points: $S_1 = (0, 2)$, $S_2 = (\sqrt{2}, 1)$ and $S_3 = (-\sqrt{2}, 1)$.

   b) For general (x, y) we find $f_{xx}(x, y) = 2y - 2$, $f_{yy}(x, y) = 2$ and $f_{xy}(x, y) = 2x = f_{yx}(x, y)$. So we find $f_{xx}(x, y)f_{yy}(x, y) - f_{xy}(x, y)f_{yx}(x, y) = 4(y - 1 - x^2)$. This implies that S_2 and S_3 are saddle points ($f_{xx}f_{yy} - f_{xy}f_{yx} < 0$) and that f has a local minimum value in S_1 ($f_{xx}f_{yy} - f_{xy}f_{yx} > 0$ and $f_{xx} > 0$).

   c) The unit vector $\mathbf{v}$ in the same direction as $\mathbf{u}$ is given by
$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}. \text{ Furthermore } \nabla f(1, 2) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$
So $D_{\mathbf{v}}(1, 2) = \mathbf{v} \bullet \nabla f(1, 2) = \frac{6}{5} + \frac{4}{5} = 2.$

3. We use the method of Lagrange multipliers. So introduce a function $L(x, y, \lambda) = e^{xy} + \lambda(x^3 + y^3 - 16)$ and find its critical points:
$$\begin{cases} 0 = \frac{\partial L}{\partial x} = ye^{xy} + 3\lambda x^2 & (A) \\ 0 = \frac{\partial L}{\partial y} = xe^{xy} + 3\lambda y^2 & (B) \\ 0 = \frac{\partial L}{\partial \lambda} = x^3 + y^3 - 16 & (C) \end{cases}$$
Multiply equation (A) by x and equation (B) by y and subtract both obtained expressions to get $3\lambda(x^3 - y^3) = 0$, with solutions $\lambda = 0$ or $x = y$. Now $\lambda = 0$ would imply that $x = 0$ and $y = 0$ which contradicts the third equation (C), so we only have $x = y$. Substitution in C gives one critical point: $(2, 2)$. The function value is e^4, which is a maximum (for example restrict both x and y between 1 and 3; then we have a closed domain where f obtains its maximum and minimum value).
There is no minimum value, since f can become arbitrary close to 0 (take $x < 0$ and $y > 0$ and let $x \to -\infty$ and $y \to \infty$ and still $x^3 + y^3 = 16$), but never becomes 0.

4. a) Make a sketch of the domain. Then you can easily verify that

$$\int_0^1 \int_{y^2}^1 \sqrt{x} e^{x^2}\, dx\, dy = \int_0^1 \int_0^{\sqrt{x}} \sqrt{x} e^{x^2}\, dy\, dx = \int_0^1 \sqrt{x} e^{x^2} \Big[y \Big]_{y=0}^{y=\sqrt{x}}\, dx$$

$$= \int_0^1 x e^{x^2}\, dx = \left[\frac{1}{2} e^{x^2} \right]_{x=0}^{x=1} = \frac{1}{2}(e-1).$$

b) Again sketch the domain. It is the part of the disc around $(0,0)$ with radius $\sqrt{2}$ and above the line $y = x$. Using $x = r\cos(\theta), y = r\sin(\theta)$ we get

$$\int\int_S \frac{1}{1+\sqrt{x^2+y^2}}\, dA = \int_{\pi/4}^{5\pi/4} \int_0^{\sqrt{2}} \frac{r}{1+r}\, dr\, d\theta = \int_{\pi/4}^{5\pi/4} \int_0^{\sqrt{2}} 1 - \frac{1}{1+r}\, dr\, d\theta$$

$$= \int_{\pi/4}^{5\pi/4} \Big[r - \ln|1+r| \Big]_{r=0}^{r=\sqrt{2}} = (\sqrt{2} - \ln(1+\sqrt{2})) \Big[\theta \Big]_{\theta=\pi/4}^{\theta=5\pi/4} = \pi(\sqrt{2} - \ln(1+\sqrt{2})).$$

5. a) $|z| = \sqrt{4+12} = 4, |w| = \sqrt{4+4} = 2\sqrt{2}$ and $\arg(z) = \frac{2}{3}\pi, \arg(w) = -\frac{1}{4}\pi$.

b) For example start with the modulus:

$$\left| \frac{z^5}{w^6} \right| = \frac{|z|^5}{|w|^6} = \frac{4^5}{(2\sqrt{2})^6} = 2.$$

And for the argument we have

$$\arg\left(\frac{z^5}{w^6}\right) = 5\arg(z) - 6\arg(w) = \frac{10}{3}\pi + \frac{3}{2}\pi = \frac{29}{6}\pi,$$

which is equivalent to $\frac{5}{6}\pi$. So

$$\frac{z^5}{w^6} = 2\left(\cos\left(\frac{5}{6}\pi\right) + i\sin\left(\frac{5}{6}\pi\right)\right) = -\sqrt{3} + i.$$

So $a = -\sqrt{3}$ and $b = 1$.

6. a) Separate the variables and find:

$$\int \frac{1}{(y+1)^2}\, dy = \int \frac{1}{x}\, dx \Longrightarrow \frac{-1}{y+1} = \ln|x| + C,$$

so the general solution is $y(x) = -1 - \frac{1}{\ln|x|+C}$. Substitute the initial value to obtain $1 = -1 - \frac{1}{C}$, so $C = -\frac{1}{2}$. So the final solution is $y(x) = -1 + \frac{1}{\frac{1}{2} - \ln|x|}$.

b) Substitute $y(x) = e^{rx}$. Then the characteristic equation becomes $r^2 - 6r + 13 = (r-3)^2 + 4 = 0$, with complex solutions $r = 3 \pm 2i$. So the general (real) solution is:

$$y(x) = c_1 e^{3x} \cos(2x) + c_2 e^{3x} \sin(2x), c_1, c_2 \in \mathbb{R}.$$