First test Calculus 2, 16 November 2015, Solutions

1. To get an idea, start with calculating the first terms of the sequence:

$$s_1 = -1,\ s_2 = -1 - \tfrac{1}{2},\ s_3 = -1 - \tfrac{1}{2} - \tfrac{1}{3},\ \text{etc.}$$

So it can easily be verified that $s_n = -\sum_{k=1}^{n} \frac{1}{k}$, which means that the sequence $\{s_n\}$ contains the (negative of the) partial sums of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. Since the harmonic series diverges, the sequence $\{s_n\}$ is diverging to $-\infty$.

2. a) Let $a_n = \dfrac{1+2\sqrt{n}}{3+4n}$ and choose $b_n = \dfrac{1}{\sqrt{n}}$. Then use the limit comparison test:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1+2\sqrt{n}}{3+4n} \div \frac{1}{\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}+2n}{3+4n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}+2}{\frac{3}{n}+4} = \frac{1}{2}.$$

Since $\sum_{n=1}^{\infty} b_n$ is divergent (p-series with $p = \frac{1}{2}$), the series $\sum_{n=1}^{\infty} \dfrac{1+2\sqrt{n}}{3+4n}$ is also divergent.

b) Remark that $n^2 \leq 2^n$ for all $n \geq 4$. Hence, for all $n \geq 4$ we have

$$\frac{n^2+2^n}{3^n} \leq \frac{2^n+2^n}{3^n} = 2\left(\frac{2}{3}\right)^n.$$

Then use the comparison test. Since $\sum_{n=1}^{n} \left(\frac{2}{3}\right)^n$ is convergent (geometric series with ratio $\frac{2}{3}$) the series $\sum_{n=1}^{\infty} \dfrac{n^2+2^n}{3^n}$ is also convergent.
[N.B. You can also use the ratio test to solve this exercise.]

3. a) We use the ratio test:

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{1}{n+1}\left(\frac{3x+1}{4}\right)^{n+1} \div \frac{1}{n}\left(\frac{3x+1}{4}\right)^{n}\right| \\ &= \left|\frac{3x+1}{4}\right| \lim_{n\to\infty} \frac{n}{n+1} = \left|\frac{3x+1}{4}\right|.\end{aligned}$$

So the series converges absolutely for $|3x+1| < 4$, that is for $-\frac{5}{3} < x < 1$, and diverges for $|3x+1| > 4$. Now determine separately the behavior in the endpoints: First take $x = 1$. We find $\sum_{n=1}^{\infty} \dfrac{1}{n}$ which is a divergent series (p-series with $p = 1$). Then consider $x = -\frac{5}{3}$. We get $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n}$. With the alternating series test we can conclude that this series converges (conditionally). So the interval of convergence is $[-\frac{5}{3}, 1)$.

b) Termwise differentiation (and using the chain rule!) yields

$$f'(x) = \sum_{n=1}^{\infty} \left(\frac{3x+1}{4}\right)^{n-1} \frac{3}{4},$$

so

$$f'(0) = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^{n-1} \frac{3}{4} = \frac{3}{4} \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{n} = \frac{3}{4} \frac{1}{1-\frac{1}{4}} = 1.$$

4. Use the well-known Maclaurin series $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$, which converges for all $t \in \mathbb{R}$. Substitute $t = -x^2$. This yields

$$f(x) = \frac{1}{x}\left(x^2 - 1 + \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}\right) = \frac{1}{x} \sum_{n=2}^{\infty} \frac{(-1)^n x^{2n}}{n!} = \sum_{n=2}^{\infty} \frac{(-1)^n x^{2n-1}}{n!},$$

converging for all $x \in \mathbb{R}$.

5. a) $\mathbf{u} \bullet \mathbf{v} = -2 + 2 - 4 = -4$ and $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} -6 \\ 2 \\ -5 \end{pmatrix} = -6\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$.

b) $\mathbf{u}_{\mathbf{v}} = \frac{\mathbf{u} \bullet \mathbf{v}}{\mathbf{v} \bullet \mathbf{v}} \mathbf{v} = -\frac{4}{9}\mathbf{v} = -\frac{8}{9}\mathbf{i} - \frac{4}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}$.

6. a) A normal vector of the plane is perpendicular to all vectors in that plane. Two of these vectors are:

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}.$$

So a normal vector is given by:

$$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \times \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}.$$

So an equation of the plane is $-2(x-1)+1(y-1)-3(z-0) = 0$, or equivalently $2x - y + 3z = 1$.

b) The distance from the point $(3, 0, -4)$ to this plane is:

$$\frac{|2 \cdot 3 - 1 \cdot 0 + 3 \cdot (-4) - 1|}{\sqrt{(-2)^2 + 1^2 + (-3)^2}} = \frac{|-7|}{\sqrt{14}} = \frac{1}{2}\sqrt{14}.$$

7. a) $\frac{\partial f}{\partial x} = 2xe^{3xy} + 3x^2ye^{3xy}$ and $\frac{\partial f}{\partial y} = 3x^3e^{3xy}$.

b) The tangent plane passes through $P = (1, 0, 1)$. Further: $\frac{\partial f}{\partial x}(1, 0) = 2$ and $\frac{\partial f}{\partial y}(1, 0) = 3$. So an equation of the tangent plane is $z = 1+2(x-1)+3(y-0) = 2x + 3y - 1$. The normal vector to the tangent plane is $\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} - 1\mathbf{k}$. So the vector notation for the normal line is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, t \in \mathbb{R},$$

or written in an alternative way as

$$\frac{x-1}{2} = \frac{y-0}{3} = \frac{z-1}{-1}.$$