VU University Amsterdam	Calculus 1
Faculty of Sciences	Second Test
Department of Mathematics	25-10-2022, 12:15-14:30 uur

The use of a calculator, the book, formula tables or lecture notes is not permitted

1. a) Since for the derivative we have $f'(x) = \frac{1}{1+(x+\ln x)^2} \cdot \left(1 + \frac{1}{x}\right) > 0$ for every $x > 0$, f is increasing on its domain and hence one-to-one. *Alternatively, use that* $x \mapsto x$, $x \mapsto \ln x$ *and* $y \mapsto \arctan y$ *are increasing.*

   b) Since $\lim_{x\to 0^+} \arctan(x + \ln x) = -\frac{\pi}{2}$, $\lim_{x\to\infty} \arctan(x + \ln x) = +\frac{\pi}{2}$, we get that $D(f^{-1}) = R(f) = (-\frac{\pi}{2}, +\frac{\pi}{2})$.

2. a) Since $f'(x) = (x + \frac{\pi}{6}) - 1 + 2\sin x \cos x$, we find that $x = -\frac{\pi}{6}$ is the (only) singular point. Since $f'(x) = -2 + 2\sin x \cos x < 0$ for $-\frac{\pi}{4} < x < -\frac{\pi}{6}$ and $f'(x) = 2\sin x \cos x$ for $-\frac{\pi}{6} < x < +\frac{\pi}{4}$, we see that $x = 0$ is the only critical point.

   b) Since the absolute maximum and minimum needs to be either a critical point, a singular point, or an endpoint, we compute $f(-\frac{\pi}{4}) = \frac{\pi}{3} + \frac{1}{2}$, $f(-\frac{\pi}{6}) = \frac{\pi}{6} + \frac{3}{4}$, $f(0) = \frac{\pi}{6}$, $f(+\frac{\pi}{4}) = \frac{\pi}{6} + \frac{1}{2}$. Hence we find the absolute minimum at $x = 0$ and the absolute maximum at $x = -\frac{\pi}{4}$.

3. Using the continuity of exp, we have $\lim\limits_{x\to\frac{\pi}{2}} \left(\sqrt{x - \frac{\pi}{2} + 1}\right)^{1/\cos x} = \exp\left(\lim\limits_{x\to\frac{\pi}{2}} \frac{\frac{1}{2}\ln\left(x - \frac{\pi}{2} + 1\right)}{\cos x}\right)$.

   Using the first l'Hopital rule, we compute $\lim\limits_{x\to\frac{\pi}{2}} \frac{\frac{1}{2}\ln\left(x - \frac{\pi}{2} + 1\right)}{\cos x} = \frac{1}{2} \lim\limits_{x\to\frac{\pi}{2}} \frac{\frac{1}{x-\frac{\pi}{2}+1}}{-\sin x} =$ $\frac{1}{2} \cdot \frac{1}{-1} = -\frac{1}{2}$. Hence we find that $\lim\limits_{x\to\frac{\pi}{2}} \left(\sqrt{x - \frac{\pi}{2} + 1}\right)^{1/\cos x} = \frac{1}{\sqrt{e}}$.

4. Since $f'(x) = \ln x + 1$ and hence $f''(x) = \frac{1}{x}$, we compute $f(e^2) = 2e^2$, $f'(e^2) = 3$, $f''(e^2) = e^{-2}$, so that the second Taylor polynomial of f about $x = e^2$ is given by

$$P_2(x) = f(e^2) + f'(e^2)(x - e^2) + \frac{f''(e^2)}{2}(x - e^2)^2 = 2e^2 + 3(x - e^2) + \frac{1}{2e^2}(x - e^2)^2.$$

5. a) Using the substitution $u = g(x) = x^2 + 1$ and hence $du = 2x\,dx$ we compute $\int_1^2 xe^{x^2+1}\,dx = \frac{1}{2}\int_{1^2+1}^{2^2+1} e^u\,du = \frac{1}{2}\left(e^5 - e^2\right)$,

b) Using partial integration twice, we compute $\int_1^2 (x^2+1)e^x\,dx = (x^2+1)e^x\Big|_1^2 - 2\int_1^2 xe^x\,dx = 5e^2 - 2e - 2\cdot xe^x\Big|_1^2 + 2\int_1^2 e^x\,dx = 5e^2 - 2e - 4e^2 + 2e + 2e^2 - 2e = 3e^2 - 2e.$

6. a) Using division of polynomials and splitting of fractions we get $\frac{x^4 - 4x^2 + 4}{(x-2)(x+2)} = x^2 + \frac{4}{(x-2)(x+2)} = x^2 + \frac{1}{x-2} - \frac{1}{x+2}$, and hence $\int \frac{x^4 - 4x^2 + 4}{(x-2)(x+2)}\,dx = \frac{1}{3}x^3 + \ln|x-2| - \ln|x+2| + c.$

b) $\int \frac{x+3}{(x+2)^2+1}\,dx = \int \frac{x+2}{(x+2)^2+1}\,dx + \int \frac{1}{(x+2)^2+1}\,dx$
$= \frac{1}{2}\ln|(x+2)^2+1| + \arctan(x+2) + c.$

7. Since

$$\int_1^{\pi/2} \frac{1+x^2}{\cos^2 x}\,dx \geq \int_1^{\pi/2} \frac{1}{\cos^2 x}\,dx = \tan x\Big|_1^{\pi/2} = \infty,$$

the statement is false.

Scoring:

1 : a) 3	2 : a) 3	3 : 4	4 : 4	5 : a) 3	6 : a) 3	7 : 3
b) 3	b) 4			b) 3	b) 3	
___	___	___	___	___	___	___
6	7	4	4	6	6	3

$$\text{Final grade} = \frac{\#\text{ points}}{4} + 1$$