VU University Amsterdam	Calculus 1
Faculty of Sciences	Second Test
Department of Mathematics	25-10-2020, 12:15-14:30

SOLUTIONS

1. Consider the function $f(x) = (x+2) \cdot e^{\frac{1}{x}}$ with domain $(2, +\infty)$.

   a) Prove that f is one-to-one.

   b) Determine the domain of f^{-1}.

   Solution.

   a) We compute the derivative of f:

$$f'(x) = e^{\frac{1}{x}} + (x+2)e^{\frac{1}{x}} \cdot \Big(-\frac{1}{x^2}\Big) = \frac{e^{\frac{1}{x}}}{x^2} \cdot (x^2 - x - 2). \quad \textbf{(1 point)}$$

   To study the sign of the derivative we determine the roots of $x^2 - x - 2 = 0$. Using the ABC formula, we get $x_- = -1$ and $x_+ = 2$. Therefore, $x^2 - x - 2 = (x+1)(x-2)$ is positive for $x > 2$ **(1 point)**. We conclude that $f'(x) > 0$ on its domain. It follows that f is increasing and hence one-to-one on $(2, +\infty)$ **(1 point)**.

   b) We have $D(f^{-1}) = R(f)$ **(1 point)**. To determine the range of f, we compute

$$\lim_{x \to 2^+} f(x) = (2+2)e^{\frac{1}{2}} = 4\sqrt{e}, \quad \lim_{x \to \infty} f(x) = \lim_{x \to \infty} (x+2) \cdot \lim_{x \to \infty} e^{\frac{1}{x}} = \infty.$$

   Therefore, $R(f) = (4\sqrt{e}, \infty)$ **(1 point)** and it follows $D(f^{-1}) = (4\sqrt{e}, \infty)$ as well.

2. Consider the function $f(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}}$ with domain $(0, +\infty)$.

   a) Find the *local* maximum and minimum values of f and determine which of them are also *absolute*.

   b) Calculate the x-value(s) of the inflection point(s) of the curve $y = f(x)$.

   Solution.

   a) Since the interval $(0, +\infty)$ has no endpoint and the function f is differentiable on its domain, local extreme points, if they exist, are among critical points, namely solutions of $f'(x) = 0$. We compute

$$f'(x) = \frac{1}{2}\frac{1}{\sqrt{x}} - \frac{1}{2}\frac{1}{x\sqrt{x}} = \frac{x-1}{2x\sqrt{x}}. \quad \textbf{(1 point)}$$

   Thus, the value $x = 1$ is the only critical point. More precisely, we have

$$f'(x) \begin{cases} < 0 & \text{for } x \in (0,1) \\ = 0 & \text{for } x = 1 \\ > 0 & \text{for } x \in (1, +\infty). \end{cases}$$

By the first derivative test, the function f attains a local minimum value at $x = 1$ **(1 point)**. Since

$$\lim_{x\to 0^+} f(x) = \infty, \qquad \lim_{x\to\infty} f(x) = \infty,$$

the function f attains an absolute minimum value at $x = 1$. **(1 point)** Alternative argument I: Since $f(x)$ is increasing for $x > 1$ and decreasing for $0 < x < 1$, then $x = 1$ is an absolute minimum. Alternative argument II: If, by contradiction f does not attain an absolute minimum at $x = 1$, then there is some $x_0 \neq 1$ for which $f(x_0) < f(1)$. Let's say $x_0 > 1$, the other case being analogous. Then the restriction of f to the interval $[1, x_0]$ attains an absolute maximum at an interior point x_1 since 1 and x_0 cannot be absolute maxima. Therefore, x_1 would be a critical point, which is impossible since 1 is the only critical point.

b) Since f is twice differentiable, inflection points, if they exist, are among the solutions to $f''(x) = 0$. We compute

$$f''(x) = \frac{1}{2}\Big(-\frac{1}{2}\frac{1}{x\sqrt{x}} + \frac{3}{2}\frac{1}{x^2\sqrt{x}}\Big) = \frac{3-x}{4x^2\sqrt{x}},$$

which vanishes for $x = 3$ **(1 point)**. Since $f''(x)$ is positive for $0 < x < 3$ and negative for $x > 3$, the function f has an inflection point for $x = 3$ **(1 point)**.

3. Calculate $\lim_{x\to 0^+} \Big(1 + \arctan(2x)\Big)^{\frac{1}{x}}$.

Solution.

The limit is an indeterminate form of the type 1^∞. Using that the exponential function is continuous and the properties of the logarithm, we can bring it to the form

$$\exp\left(\lim_{x\to 0^+} \frac{\ln(1+\arctan(2x))}{x}\right). \quad \textbf{(1 point)}$$

Since the limit inside the exponential function is of type $\frac{0}{0}$, we can use l'Hôpital rule **(1 point)**. We find

$$\lim_{x\to 0^+} \frac{\ln(1+\arctan(2x))}{x} = \lim_{x\to 0^+} \frac{\frac{1}{1+\arctan(2x)} \cdot \frac{1}{1+(2x)^2} \cdot 2}{1} = 2. \quad \textbf{(1 point)}$$

Therefore,

$$\lim_{x\to 0^+} \Big(1 + \arctan(2x)\Big)^{\frac{1}{x}} = \exp\left(\lim_{x\to 0^+} \frac{\ln(1+\arctan(2x))}{x}\right) = e^2. \quad \textbf{(1 point)}$$

4. Consider the function $f(x) = \ln(\cos x)$.

   a) Find the linearization $L(x)$ of $f(x)$ about $x_0 = \frac{\pi}{4}$ and use it to give an approximate value of $\ln(\cos(\frac{\pi}{5}))$.

   b) If $E_1(\frac{\pi}{5})$ denotes the resulting error, show that

$$\Big|E_1\Big(\frac{\pi}{5}\Big)\Big| < \Big(\frac{\pi}{20}\Big)^2.$$

Solution.

a) The linearization of f about $x_0 = \frac{\pi}{4}$ is

$$L(x) = f(\tfrac{\pi}{4}) + f'(\tfrac{\pi}{4})(x - \tfrac{\pi}{4}) \quad \textbf{(1 point)}.$$

We compute $f'(x) = \frac{1}{\cos x} \cdot (-\sin x) = -\tan x$ **(1 point)** and, using that $\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$ and $\tan(\frac{\pi}{4}) = 1$, we get

$$L(x) = \ln(\tfrac{1}{\sqrt{2}}) - (x - \tfrac{\pi}{4}) \quad \textbf{(1 point)}.$$

Therefore, $\ln(\cos\frac{\pi}{5}) \approx L(\frac{\pi}{5}) = \ln(\frac{1}{\sqrt{2}}) + \frac{\pi}{20}$ **(1 point)**.

b) The error formula reads

$$E_1(x) = \frac{f''(c)}{2}(x - \tfrac{\pi}{4})^2 \quad \textbf{(1 point)}$$

for some $c \in (\frac{\pi}{5}, \frac{\pi}{4})$. Since $f''(x) = -\frac{1}{\cos^2 x}$, we have

$$\left| E_1\left(\frac{\pi}{5}\right) \right| = \frac{1}{2\cos^2 c}\left(\frac{\pi}{20}\right)^2. \quad \textbf{(1 point)}$$

Since $\cos c$ is decreasing for $c \in (\frac{\pi}{5}, \frac{\pi}{4})$, we get $\frac{1}{2\cos^2 c} < \frac{1}{2\cos^2(\frac{\pi}{4})} = 1$ **(1 point)**. Inserting this inequality in the formula for the error, we get

$$\left| E_1\left(\frac{\pi}{5}\right) \right| < \left(\frac{\pi}{20}\right)^2.$$

5. Compute

a) $\displaystyle\int_0^2 2e^{-x^2} x^3\, dx$,

b) $\displaystyle\int_1^{e^2} \frac{\ln x}{2\sqrt{x}}\, dx$.

Solution.

a) Making the substitution $u = g(x) := x^2$ so that $g'(x) = 2x$ **(1 point)**, we get

$$\int_0^2 2e^{-x^2} x^3\, dx = \int_0^2 e^{-g(x)} g(x) \cdot g'(x)\, dx = \int_0^4 e^{-u} u\, du. \quad \textbf{(1 point)}$$

Integrating by parts we get

$$\int_0^4 ue^{-u}\, du \overset{\textbf{(1pt)}}{=} -e^{-u}u\Big|_0^4 + \int_0^4 e^{-u}\, du = e^{-u}(-u-1)\Big|_0^4 \overset{\textbf{(1pt)}}{=} 1 - 5e^{-4}.$$

Alternative argument: Compute first indefinite integral and then substitute the endpoints of the interval at the end.

b) We integrate by parts

$$\int_1^{e^2} \frac{\ln x}{2\sqrt{x}}\, dx = \sqrt{x}\ln x\Big|_1^{e^2} - \int_1^{e^2} \sqrt{x}\frac{1}{x}\, dx = 2e - \int_1^{e^2} \frac{1}{\sqrt{x}}\, dx, \quad \textbf{(1 point)}$$

which is equal to $2e - \left(2\sqrt{x}\Big|_1^{e^2}\right) = 2$ **(1 point)**.

6. Calculate

a) $\displaystyle\int \frac{1-x}{(x+2)^2+2(x+2)+2}\,dx$,

b) $\displaystyle\int \frac{x^3+2}{x^2-x}\,dx$.

Solution.

a) Completing the square, we have $(x+2)^2+2(x+2)+2 = x^2+6x+10 = (x+3)^2+1$ **(1 point)**. Therefore, using that $1-x = -(x+3)+4$, we get

$$\int \frac{1-x}{(x+2)^2+2(x+2)+2}\,dx = \int \frac{1-x}{(x+3)^2+1}\,dx$$
$$= -\int \frac{x+3}{(x+3)^2+1}\,dx + 4\int \frac{1}{(x+3)^2+1}\,dx \textbf{ (1 point)}.$$

Using the integration formulas from the lectures, we find

$$\int \frac{1-x}{(x+2)^2+2(x+2)+2}\,dx = -\frac{1}{2}\ln\left((x+3)^2+1\right)+4\arctan(x+3)+C. \textbf{ (1 point)}$$

b) Since the numerator has higher degree than the denominator, we do long division and find

$$x^3+2 = (x^2-x)(x+1)+x+2.$$

Therefore,

$$\int \frac{x^3+2}{x^2-x}\,dx = \int (x+1)\,dx + \int \frac{x+2}{x^2-x}\,dx = \frac{x^2}{2}+x+\int \frac{x+2}{x^2-x}\,dx \textbf{ (1 point)}.$$

Since $x^2-x = x(x-1)$ has two distinct real roots, we write the partial fraction decomposition

$$\frac{x+2}{x(x-1)} = \frac{3}{x-1}+\frac{-2}{x} \quad \textbf{(1 point)},$$

where the two coefficients are found by $\frac{1+2}{1} = 3$ and $\frac{0+2}{0-1} = -2$. Alternatively the coefficients A and B are given by solving the system $A+B=1$, $-B=2$. Therefore,

$$\int \frac{x+2}{x(x-1)}\,dx = 3\int \frac{1}{x-1}\,dx - 2\int \frac{1}{x}\,dx = 3\ln|x-1|-2\ln|x|+C. \quad \textbf{(1 point)}$$

Putting all together, we get

$$\int \frac{x^3+2}{x^2-x}\,dx = \frac{x^2}{2}+x+3\ln|x-1|-2\ln|x|+C.$$

7. Is the following statement true or false? Motivate your answer.

$$\int_1^2 \frac{e^{-x^2}}{2\sqrt{x-1}}\,dx = \infty.$$

Solution. We have $e^{-x^2} \le 1$ since $-x^2 \le 0$ **(1 point)**. Therefore,

$$\int_1^2 \frac{e^{-x^2}}{2\sqrt{x-1}}\,dx \le \int_1^2 \frac{1}{2\sqrt{x-1}}\,dx \overset{\textbf{(1pt)}}{=} \lim_{r\to 1^+}\left(\sqrt{2-1}-\sqrt{r-1}\right) \overset{\textbf{(1pt)}}{=} 1$$

and the statement is false.

Scoring:

1	2	3	4	5	6	7
1 : a) 3	2 : a) 3	3 : 4	4 : a) 4	5 : a) 4	6 : a) 3	7 : 3
b) 2	b) 2		b) 3	b) 2	b) 3	
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5	5	4	7	6	6	3

$$\text{Final grade} = \frac{\#\text{ points}}{4} + 1$$