Exercise 1. Consider the function f defined by $f(x) = \sin\left(\frac{\pi}{x+2}\right)$ with domain $(0, \infty)$.

a) Prove that f is one-to-one.

b) Determine the domain of f^{-1}.

Solution: First, we want to show that f is one-to-one. The first derivative yields:

$$f'(x) \overset{(\mathbf{1P})}{=} -\underbrace{\frac{\pi}{(x+2)^2}}_{>0}\ \underbrace{\cos\left(\frac{\pi}{x+2}\right)}_{>0} \overset{(\mathbf{1P})}{<} 0.$$

The second inequality holds because $\frac{\pi}{x+2} \in (0, \frac{\pi}{2})$ for all $x \in (0, \infty)$. Hence, f is decreasing on its domain **(1P)** and therefore f is one-to-one.

As for b), remember that the domain of the inverse coincides with the range of the function, i.e.:

$$\operatorname{dom}(f^{-1}) \overset{(\mathbf{1P})}{=} \operatorname{range}(f) = (0, 1),$$

since $\sin\left(\frac{\pi}{0+2}\right) = 1$ **(1P)** and $\lim_{x\to\infty} \sin\left(\frac{\pi}{x+2}\right) = \sin 0 = 0$ **(1P)**.

Exercise 2. The function f is defined by

$$f(x) = \mathrm{e}^{-x}\left(x^2 - 2x - 3\right).$$

a) Find the maxima and minima of f and classify them as local or absolute.

b) Calculate the x-values of the inflection point(s) of the curve $y = f(x)$.

Solution: We start with a). To find the local minima/maxima, we have to check the necessary condition first:

$$0 \overset{!}{=} f'(x) \overset{(\mathbf{1P})}{=} -\mathrm{e}^{-x}\left(x^2 - 4x - 1\right)$$

Since the exponential function never crosses zero, it is sufficient to find the roots of the polynomial. Using the ABC-Formula, we obtain $x_\pm = 2 \pm \sqrt{5}$. **(1P)** Since $f'(x) < 0$ for $x < 2 - \sqrt{5}$ and $x > 2 + \sqrt{5}$, and $f'(x) > 0$ for $2 - \sqrt{5} < x < 2 + \sqrt{5}$, we find by the First Derivative Test that $(2 - \sqrt{5}, f(2 - \sqrt{5}))$ is a minimum and $(2 + \sqrt{5}, f(2 + \sqrt{5}))$ is a maximum. **(1P)** Since $\lim_{x\to-\infty} f(x) = \infty$, x_+ cannot be a global maximum. On the other hand, since $\lim_{x\to\infty} f(x) = 0$ and $f(x_-) = \mathrm{e}^{-x_-}\left(4 - 3\sqrt{5}\right) < 0$, we conclude that x_- must be a global minimum. **(1P)**

For part b), we want to find the inflection points of the graph $y = f(x)$. Any inflection point x must satisfy:

$$f''(x) \overset{\textbf{(1P)}}{=} \mathrm{e}^{-x}\left(x^2 - 6x + 3\right) = 0.$$

From the second derivative we infer $x_{\pm} = 3 \pm \sqrt{6}$ as possible inflection points. **(1P)** Since in both points the sign of $f''(x)$ changes, we know that both points $x_{\pm}$ are inflection points. **(1P)**

Exercise 3. Calculate $\lim_{x \to 0^+}\left(\frac{1}{x^2} - \frac{1}{\ln(x+1)}\right)$.

Solution: Reformulate the expression in order to apply L'Hôspital:

$$\lim_{x \to 0^+}\left(\frac{1}{x^2} - \frac{1}{\ln(x+1)}\right) \overset{\textbf{(1P)}}{=} \lim_{x \to 0^+}\left(\frac{\ln(x+1) - x^2}{x^2 \ln(x+1)}\right)$$

Now, we can apply L'Hôspital's rule **(1P)** since the limit is of the form "$\frac{0}{0}$":

$$\lim_{x \to 0^+}\left(\frac{\ln(x+1) - x^2}{x^2 \ln(x+1)}\right) \overset{\textbf{(1P)}}{=} \lim_{x \to 0^+} \frac{\frac{1}{x+1} - 2x}{2x\ln(x+1) + \frac{x^2}{x+1}}$$

The numerator tends to one while the denominator tends to zero. But since we approach from the right, the denominator is always positive and the whole expression tends to $+\infty$ **(1P)**.

Exercise 4. Find $P_2(x)$, the second Taylor polynomial of $f(x) = \sin^{-1} x$ about $x = \frac{\sqrt{3}}{2}$.

Solution: In order to obtain the second Taylor polynomial, we require the first two derivatives of f:

$$f'(x) = \frac{1}{\sqrt{1 - x^2}},\ f''(x) = \frac{x}{(1 - x^2)^{\frac{3}{2}}}.$$

Now, we have to evaluate these at the expansion point:

$$f\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3},\ f'\left(\frac{\sqrt{3}}{2}\right) = 2,\ f''\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}.\ \textbf{(1P)} + \textbf{(1P)} + \textbf{(1P)}$$

The Taylor polynomial is hence given by:

$$\begin{aligned} P_2(x) &= f\left(\frac{\sqrt{3}}{2}\right) + f'\left(\frac{\sqrt{3}}{2}\right)\left(x - \frac{\sqrt{3}}{2}\right) + \frac{f''\left(\frac{\sqrt{3}}{2}\right)}{2!}\left(x - \frac{\sqrt{3}}{2}\right)^2 \\ &= 2\sqrt{3}\left(x - \frac{\sqrt{3}}{2}\right)^2 + 2\left(x - \frac{\sqrt{3}}{2}\right) + \frac{\pi}{3}.\ \textbf{(1P)} \end{aligned}$$

Exercise 5. Compute

a) $\int_0^{\frac{\pi}{6}} \sin^2(x)\cos^3(x)\,\mathrm{d}x$,

b) $\int_{-\pi}^{\pi} x^2 \cos x\,\mathrm{d}x$.

Solution: We start by solving the integral in a):

$$\int_0^{\frac{\pi}{6}} \sin^2(x)\cos^3(x)\,\mathrm{d}x = \int_0^{\frac{\pi}{6}} \sin^2(x)\left(1-\sin^2(x)\right)\cos(x)\,\mathrm{d}x$$
$$\overset{(\mathbf{1P})}{=} \int_0^{\frac{1}{2}} u^2\left(1-u^2\right)\,\mathrm{d}u \overset{(\mathbf{1P})}{=} \left(\frac{u^3}{3}-\frac{u^5}{5}\right)\Bigg|_0^{\frac{1}{2}} \overset{(\mathbf{1P})}{=} \frac{1}{2^3\cdot 3}-\frac{1}{2^5\cdot 5}\left(=\frac{17}{480}\right),$$

where we substituted $u=\sin(x)$ and $\mathrm{d}u=\cos(x)\,\mathrm{d}x$ accordingly.

The integral b) can be solved by using integration by parts twice:

$$\int_{-\pi}^{\pi} x^2\cos x\,\mathrm{d}x \overset{(\mathbf{1P})}{=} \underbrace{x^2\sin(x)\Big|_{-\pi}^{\pi}}_{=0} -2\int_{-\pi}^{\pi} x\sin(x)\,\mathrm{d}x$$
$$\overset{(\mathbf{1P})}{=} \underbrace{2x\cos(x)|_{-\pi}^{\pi}}_{=-4\pi} - \underbrace{2\int_{-\pi}^{\pi}\cos(x)\,\mathrm{d}x}_{=0} \overset{(\mathbf{1P})}{=} -4\pi.$$

Exercise 6. Calculate

a) $\int \frac{x^2+3}{x(x+3)}\,\mathrm{d}x$,

b) $\int \frac{x}{x^2-2x+2}\,\mathrm{d}x$.

Solution: We want to apply partial fraction decomposition to the integral a). But before, we have to reformulate the integral in the appropriate manner:

$$\int \frac{x^2+3}{x(x+3)}\,\mathrm{d}x \overset{(\mathbf{1P})}{=} \underbrace{\int 1\,\mathrm{d}x}_{=x} - \int \frac{3x-3}{x^2+3x}\,\mathrm{d}x.$$

Now, we have the following ansatz:

$$\frac{3x-3}{x^2+3x} = \frac{A}{x}+\frac{B}{x+3}.$$

Due to the lecture, we can solve for A and B using the following trick:

$$A = \lim_{x\to 0} \cancel{x} \frac{3x-3}{\cancel{x}(x+3)} = -1,\ B = \lim_{x\to -3} \cancel{(x+3)} \frac{3x-3}{x\cancel{(x+3)}} = 4.$$

Therefore, the result is given by:

$$\int \frac{x^2+3}{x(x+3)}\,\mathrm{d}x = x + \ln|x| - 4\ln|x+3| + c.\,(\mathbf{1P}) + (\mathbf{1P})$$

The integral in b) can be solved by completing the square:

$$\int \frac{x}{x^2-2x+2}\,\mathrm{d}x \overset{(\mathbf{1P})}{=} \underbrace{\int \frac{x-1}{(x-1)^2+1}\,\mathrm{d}x}_{=:I_1} + \underbrace{\int \frac{1}{(x-1)^2+1}\,\mathrm{d}x}_{=:I_2}.$$

From the lecture and the book we know that

$$I_1 = \frac{1}{2}\ln\left|(x-1)^2+1\right|.\,(\mathbf{1P})$$
$$I_2 = \tan^{-1}(x-1).\,(\mathbf{1P})$$

Thus, we get:

$$\int \frac{x}{x^2-2x+2}\,\mathrm{d}x = \frac{1}{2}\ln\left|(x-1)^2+1\right| + \tan^{-1}(x-1) + c.$$

Exercise 7. Is the following statement true or false? Motivate your answer.

$$\int_1^\infty \frac{2+\sin(x^2)}{x}\,\mathrm{d}x = \infty.$$

Solution: The statement is true. This can be seen as follows. Since

$$\frac{2+\sin(x^2)}{x} \geq \frac{1}{x}$$

for all $x \geq 1$, we have

$$\int_1^\infty \frac{2+\sin(x^2)}{x}\,\mathrm{d}x \overset{(\mathbf{1P})}{\geq} \int_1^\infty \frac{1}{x}\,\mathrm{d}x \overset{(\mathbf{1P})}{=} \lim_{R\to\infty} \ln R - \ln 1 \overset{(\mathbf{1P})}{=} \infty.$$