Exercise 1. Consider the polynomial

$$p(x) = x^3 - 7x + 6.$$

Show that $x - 2$ is a factor of p, and then find all roots of p.

Solution: In order to show that $x - 2$ is a factor of $p(x)$, it is enough to show that 2 is a root of $p(x)$. Indeed:

$$p(2) = 2^3 - 7 \cdot 2 + 6 = 0. \ (\mathbf{1P})$$

Thus, $p(x) = (x-2)q(x)$ for some polynomial $q(x)$. Now, determine $q(x)$ by long division to obtain $q(x) = x^2 + 2x - 3$ **(1P)**. To find the extant roots, we simply apply the *ABC* formula **(1P)** to $q(x)$ and get:

$$x_{1,2} = -1 \pm 2. \ (\mathbf{1P})$$

Exercise 2. Calculate the following limits, or explain why they do not exist:

a)
$$\lim_{x\to\infty} \frac{\sqrt{6x^2 - 4x + 7}}{|3x + 2|},$$

b)
$$\lim_{x\to 0} \frac{x \sin x}{\sqrt{1 + x^2} - \sqrt{1 - x^2}},$$

c)
$$\lim_{x\to 2} \left(2\lfloor x \rfloor - 1\right).$$

Solution:

a) We pull out the highest power of x:

$$\lim_{x\to\infty} \frac{\sqrt{6x^2 - 4x + 7}}{|3x + 2|} \overset{(\mathbf{1P})}{=} \lim_{x\to\infty} \frac{\sqrt{x^2(6 - \frac{4}{x} + \frac{7}{x^2})}}{\left|x(3 + \frac{2}{x})\right|} \overset{(\mathbf{1P})}{=} \lim_{x\to\infty} \frac{\cancel{x}}{\cancel{x}} \cdot \frac{\sqrt{6 - \frac{4}{x} + \frac{7}{x^2}}}{\left|3 + \frac{2}{x}\right|} \overset{(\mathbf{1P})}{=} \sqrt{\frac{2}{3}}$$

b) We start by expanding the fraction using the third binomial rule:

$$\begin{aligned}
\lim_{x\to 0}\frac{x\sin x}{\sqrt{1+x^2}-\sqrt{1-x^2}} &\overset{(\mathbf{1P})}{=} \lim_{x\to 0}\frac{x\sin x\cdot\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}{\left(\sqrt{1+x^2}-\sqrt{1-x^2}\right)\cdot\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)} \\
&\overset{(\mathbf{1P})}{=} \lim_{x\to 0}\left[\frac{x\sin x}{2x^2}\cdot\underbrace{\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}_{\xrightarrow{x\to 0}2}\right] \\
&\overset{(\mathbf{1P})}{=} \lim_{x\to 0}\frac{\sin x}{x} \\
&\overset{(\mathbf{1P})}{=} 1
\end{aligned}$$

Here, the latter limit is known from the lecture.

c) The limit does not exist since the left and the right limits are different. To see that, we have to take a closer look at the floor function $\lfloor x\rfloor$ which returns the integer part of x. It is readily checked (and also discussed in the book) that the floor function is right-continuous but not left-continuous. That given, our left limit becomes

$$\begin{aligned}
\lim_{x\to 2^-}\left(2\lfloor x\rfloor-1\right) &= 2\lfloor 1\rfloor-1 \\
&= 1.\ (\mathbf{1P})
\end{aligned}$$

Whereas the right limit yields

$$\begin{aligned}
\lim_{x\to 2^+}\left(2\lfloor x\rfloor-1\right) &= 2\lfloor 2\rfloor-1 \\
&= 3.\ (\mathbf{1P})
\end{aligned}$$

Thus, the left and right limits are different and the overall limit does not exist. **(1P)**

Exercise 3. For which real numbers a and b is the function

$$f(x)=\begin{cases} a\cos\left(x+\frac{\pi}{3}\right) & x\le 0, \\ x^2+bx+1 & x>0, \end{cases}$$

a) continuous at $x=0$?

b) differentiable at $x=0$?

Solution: We start with a). The function f is continuous at $x = 0$ if

$$\lim_{x \to 0} f(x) = f(0). \ \mathbf{(1P)}$$

As for the left limit, there is nothing to check since $a \cos\left(x + \frac{\pi}{3}\right)$ is already known to be continuous. Its value is given by inserting $x = 0$ right away:

$$f(0) = a \cos\left(\frac{\pi}{3}\right) = \frac{a}{2}. \ \mathbf{(1P)}$$

For the right limit, we get:

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} \left(x^2 + bx + 1\right) \\ &= 1. \end{aligned}$$

For continuity, we require $f(0) = 1$. Therefore $a = 2$ and the number b can attain any value. **(1P)**

Let us proceed with b). Since every differentiable function must be also continuous, condition a) must already be satisfied, and we can (and must) set $a = 2$. The function f is differentiable at $x = 0$ if the differential quotient

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

exists. In particular, the limits from both sides must be equal. **(1P)** The limit from the left appears to be the differential quotient for the function $2\cos\left(x + \frac{\pi}{3}\right)$ which is swiftly calculated using differentiation rules:

$$\frac{\mathrm{d}}{\mathrm{d}x} 2\cos\left(x + \frac{\pi}{3}\right) = -2\sin\left(x + \frac{\pi}{3}\right).$$

This evaluated at $x = 0$ yields

$$\begin{aligned} \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} &= -2\sin\left(\frac{\pi}{3}\right) \\ &= -\sqrt{3}. \ \mathbf{(1P)} \end{aligned}$$

To obtain the right limit, let us plug in all necessary values:

$$\begin{aligned} \lim_{h \to 0^+} \frac{f(h) - f(0)}{h} &= \lim_{h \to 0^+} \frac{h^2 + bh + \cancel{1} - \cancel{1}}{h} \\ &= b. \end{aligned}$$

Since we require both limits to be the same, we have $a = 2$ and $b = -\sqrt{3}$. **(1P)**

Exercise 4. Prove that the equation

$$\tan x + x^3 - \frac{1}{2} = 0$$

has exactly one solution in $[0, \frac{\pi}{4}]$ by showing that

a) it has at least one solution in $[0, \frac{\pi}{4}]$,

b) it has at most one solution in $[0, \frac{\pi}{4}]$.

Solution: Let us start with a). We define $f(x) = \tan(x) + x^3 - \frac{1}{2}$. At the boundary, we obtain the following values:

$$f(0) = -\frac{1}{2} < 0, \quad f(\frac{\pi}{4}) = 1 + \frac{\pi^3}{64} - \frac{1}{2} > 0. \ (\mathbf{1P})$$

Since f is continuous, we conclude from the Intermediate Value Theorem **(1P)** that f must have at least one zero in $[0, \frac{\pi}{4}]$.

To show b), that is f has at most one solution in $[0, \frac{\pi}{4}]$, we go by contradiction. Assume there are at least two distinct zeros $a, b \in [0, \frac{\pi}{4}]$, i.e.

$$f(a) = 0 = f(b).$$

Since f is differentiable in $(0, \frac{\pi}{4})$, we conclude by the theorem of Rolle **(1P)** the existence of an $c \in (a, b)$ such that $f'(c) = 0$. **(1P)** But this contradicts the calculation:

$$f'(c) = \frac{1}{\cos^2 c} + 2c^2 > 0 \quad \text{for all } c \in \left(0, \frac{\pi}{4}\right). \ (\mathbf{1P})$$

Exercise 5. Consider the graph of the equation

$$3y^2 = x^2 - 2xy.$$

a) Calculate $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.

b) Write down the equation for the tangent line to the graph in the point $(3, 1)$.

Solution: To facilitate calculations, we reformulate the equation:

$$3y^2 - x^2 + 2xy = 0. \tag{1}$$

As for a), we assume that the set of all points given by (1) can locally be written in terms of a function $y(x)$. Then, by the chain rule, we obtain:

$$\begin{aligned} 0 &\overset{(\mathbf{1P})}{=} \frac{\mathrm{d}}{\mathrm{d}x}\left(3y^2 - x^2 + 2xy\right) \\ &= 6y'y - 2x + 2y + 2xy'. \ (\mathbf{1P} + \mathbf{1P}) \end{aligned}$$

Solving this for $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$, we finally end up with

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x - 2y}{6y + 2x}. \ (\mathbf{1P})$$

In the second part b), let us recall the general formula for the tangent line:

$$y = m(x - x_0) + y_0.$$

We already know $(x_0, y_0) = (3, 1)$ from the assumptions in the exercise. The slope m is simply given by $\frac{\mathrm{d}y}{\mathrm{d}x}$ evaluated at that point:

$$m = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(3,1)} = \frac{1}{3}. \ (\mathbf{1P})$$

We conclude

$$y = \frac{1}{3}(x - 3) + 1 = \frac{x}{3}. \ (\mathbf{1P})$$

Exercise 6. Prove, using the mean value theorem, that for all $0 \leq x \leq \frac{1}{4}$:

$$2\sqrt{x} - \sin x \geq x.$$

Solution: Recall the mean value theorem: if a function f is differentiable in the interval (a, b) and continuous on $[a, b]$, then there is an $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \ (\mathbf{1P})$$

First of all, we have to choose f properly. It turns out that

$$f(x) = 2\sqrt{x} - \sin x$$

is a suitable choice. This function is certainly differentiable in $(0, \frac{1}{4})$ and moreover continuous in $[0, \frac{1}{4}]$. **(1P)** Now, suppose $x \in \left(0, \frac{1}{4}\right]$, then we get

$$\begin{aligned} \frac{2\sqrt{x} - \sin x}{x} &= \frac{f(x) - f(0)}{x} \\ &= f'(c) \\ &= \frac{1}{\sqrt{c}} - \cos c \ (\mathbf{1P}) \end{aligned} \tag{2}$$

for an $c \in (0, x)$. Notice that we have used

$$\begin{aligned} f(0) &= 2\sqrt{0} - \sin 0 \\ &= 0 \end{aligned}$$

Since $0 < c < x < \frac{1}{4}$, we can estimate

$$\begin{aligned} \frac{1}{\sqrt{c}} - \cos c &> 2 - \cos c \\ &\geq 1, \ (\mathbf{1P}) \end{aligned}$$

because $|\cos c| \leq 1$. The above inequality (2) holds true for any $x \in \left(0, \frac{1}{4}\right]$ and hence we immediately get for those x:

$$2\sqrt{x} - \sin x > x.$$

As for $x = 0$, we obtain equality and therefore

$$2\sqrt{x} - \sin x \geq x$$

as desired.