Vrije Universiteit Amsterdam	Calculus 1, Second Test
Faculty of Sciences	21-10-2019
Department of Mathematics	12.00 - 2.00 pm

1. a) $\lim_{x\to 0+} f(x) = 0$ **(1P)**, $\lim_{x\to\infty} f(x) = \infty$ **(1P)**.

   b) $f'(x) = x^2(3\ln x - 2) = 0$ if $x = e^{\frac{2}{3}}$ **(1P)** which sits in $[1, e]$. Since $f(1) = -1$, $f(e) = 0$ and $f(e^{\frac{2}{3}}) = -\frac{1}{3}e^2 < -1$, $x = e^{\frac{2}{3}}$ is absolute minimum **(1P)** and $x = e$ is absolute maximum **(1P)**.

   c) $f''(x) = x(6\ln x - 1)$ **(1P)** $\Rightarrow f''(x)$ changes sign at $x = e^{\frac{1}{6}} \Rightarrow f$ has an inflection point at $x = e^{\frac{1}{6}}$ **(1P)**.

2. $f'(x) = 5x^4 + \dfrac{1}{x^2+1} > 0 \Rightarrow f$ is increasing **(1P)** $\Rightarrow f$ is one-to-one and hence f^{-1} exists **(1P)**. Furthermore, $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to+\infty} f(x) = +\infty \Rightarrow$ domain of f^{-1} = range of $f = (-\infty, +\infty)$ **(1P)**.

3. $\lim_{x\to 0}(\cos x)^{x^{-2}} = e^{\lim_{x\to 0}\frac{\ln\cos x}{x^2}}$ **(1P)** with $\lim_{x\to 0}\dfrac{\ln\cos x}{x^2}$ by l'Hopital being equal to $\lim_{x\to 0}\dfrac{-\tan x}{2x}$ **(1P)** which again by l'Hopital is equal to $\lim_{x\to 0}\dfrac{-\frac{1}{\cos^2 x}}{2} = -\dfrac{1}{2}$ **(1P)**, so that $\lim_{x\to 0}(\cos x)^{x^{-2}} = e^{-\frac{1}{2}}$ **(1P)**.

4. a) $f(\frac{\pi}{4}) = \tan\frac{\pi}{4} = 1$ **(1P)**, $f'(\frac{\pi}{4}) = \dfrac{1}{\cos^2\frac{\pi}{4}} = 2$ **(1P)**
   $\Rightarrow L(x) = f(\frac{\pi}{4}) + f'(\frac{\pi}{4})(x - \frac{\pi}{4}) = 2(x - \frac{\pi}{4}) + 1$ **(1P)**.

   b) $\tan\frac{\pi}{5} \approx L(\frac{\pi}{5})$ **(1P)** with $L(\frac{\pi}{5}) = 2(\frac{\pi}{5} - \frac{\pi}{4}) + 1 = 1 - \frac{\pi}{10}$ **(1P)**.

   c) $E(\frac{\pi}{5}) = \frac{1}{2}f''(s)\cdot(\frac{\pi}{5} - \frac{\pi}{4})^2$ for some s in $(\frac{\pi}{5}, \frac{\pi}{4})$ **(1P)**, so that $|E(\frac{\pi}{5})| \le \frac{1}{2}\cdot\dfrac{2\tan\frac{\pi}{4}}{\cos^2\frac{\pi}{4}}\cdot\dfrac{\pi^2}{400} = \dfrac{\pi^2}{200} < \dfrac{1}{10}$ **(1P)**.

5. a) $\int_1^{e^2} \frac{\ln\sqrt{x}}{\sqrt{x}}\,dx = 2\int_1^e \ln u\,du$ using the substitution $u = \sqrt{x}$ **(1P)**,

and $2\int_1^e \ln u\,du = 2u(\ln u - 1)|_1^e = 2$ **(1P)**.

b) $\int_1^{e^2} \frac{\ln\sqrt{x}}{\sqrt{x}}\,dx = (\ln\sqrt{x}\cdot 2\sqrt{x})|_1^{e^2} - \int_1^{e^2} \frac{1}{2x}\cdot 2\sqrt{x}\,dx$ **(1P)**

which is equal to $(\ln\sqrt{x}\cdot 2\sqrt{x})|_1^{e^2} - 2\sqrt{x}|_1^{e^2} = 2$ **(1P)**.

6. a) $\int e^x \cos x\,dx = \frac{be^{ax}\sin bx + ae^{ax}\cos bx}{b^2+a^2} + c$ with $a = b = 1$ **(1P)**,

which gives $\frac{1}{2}e^x(\sin x + \cos x) + c$ **(1P)**. **Or:** $\int e^x\cos x\,dx = e^x \sin x - \int e^x \sin x\,dx = e^x\sin x + e^x\cos x - \int e^x\cos x\,dx$ **(1P)** which again gives $\frac{1}{2}e^x(\sin x + \cos x) + c$ **(1P)**.

b) Since $x^2 - x - 6 = (x+2)(x-3)$ we make the ansatz $\frac{8x-14}{(x^2-x-6)(x-1)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3}$ **(1P)** and compute $A = 1$, $B = -2$ and $C = 1$ **(1P)**.

Hence $\int \frac{8x-14}{(x^2-x-6)(x-1)}\,dx = \ln|x-1| - 2\ln|x+2| + \ln|x-3| + c$ **(1P)**.

c) $\int \frac{x-1}{x^2-4x+5}\,dx = \int \frac{(x-2)+1}{(x-2)^2+1}\,dx$ **(1P)**, which is equal to $\int \frac{1}{(x-2)^2+1}\,dx + \int \frac{(x-2)}{(x-2)^2+1}\,dx$ **(1P)**, which gives $\tan^{-1}(x-2) + \frac{1}{2}\ln((x-2)^2+1) + c$ **(1P)**.

7. Since $e^x\tan x \geq \tan x \geq 0$ for x in $(0, \frac{\pi}{2})$ we have $\int_0^{\frac{\pi}{2}} e^x \tan x\,dx \geq \int_0^{\frac{\pi}{2}} \tan x\,dx$ **(1P)**

with $\int_0^{\frac{\pi}{2}} \tan x\,dx = -\ln\cos x\Big|_0^{\frac{\pi}{2}}$ **(1P)** with $-\ln\cos x\Big|_0^{\frac{\pi}{2}} = +\infty$ **(1P)**.

Scoring:

1	2	3	4	5	6	7
1 : a) 2	2 : 3	3 : 4	4 : a) 3	5 : a) 2	6 : a) 2	7 : 3
b) 3			b) 2	b) 2	b) 3	
c) 2			c) 2		c) 3	
7	3	4	7	4	8	3

$$\text{Final grade} = \frac{\#\text{ points}}{4} + 1$$