# Solutions First Test Calculus 1 BA 2019 

1a) $\dfrac{6}{x-1} > x \iff \dfrac{x(x-1)}{x-1} - \dfrac{6}{x-1} < 0$

$\iff \dfrac{x^2-x-6}{x-1} < 0$. Solving $x^2-x-6=0$

gives $x_1=-2$ and $x_2=+3$, so that the inequality is equivalent to $\dfrac{(x+2)(x-3)}{x-1} < 0$.

| $x$ | | $-2$ | | $1$ | | $3$ | |
|---|---|---|---|---|---|---|---|
| $x+2$ | $-$ | $0$ | $+$ | $+$ | $+$ | $+$ | $+$ |
| $x-1$ | $-$ | $-$ | $-$ | $0$ | $+$ | $+$ | $+$ |
| $x-3$ | $-$ | $-$ | $-$ | $-$ | $-$ | $0$ | $+$ |
| | $-$ | $0$ | $+$ | $0$ | $-$ | $0$ | $+$ |

Solution set: $\boxed{(-\infty,-2) \cup (1,3)}$

1b) $|2x-3| \geq |x+3| \iff |2x-3| - |x+3| \geq 0$

$\iff \dfrac{(2x-3)^2-(x+3)^2}{|2x-3|+|x+3|} \geq 0 \iff (2x-3)^2-(x+3)^2 \geq 0$

$\iff 4x^2-12x+9-x^2-6x-9 \geq 0 \iff 3x^2-18x \geq 0$

$\iff 3x(x-6) \geq 0 \Rightarrow$ solution set: $\boxed{(-\infty,0] \cup [6,+\infty)}$

②

2a) Since $x^3+8<0$ for all $x<-2$ we have

$$\lim_{x\to-\infty} \frac{2x^3-4x+9}{|x^3+8|} = \lim_{x\to-\infty} \frac{2x^3-4x+9}{-x^3-8}$$

$$= \lim_{x\to-\infty} \frac{x^3\left(2-4\cdot\frac{1}{x^2}+9\cdot\frac{1}{x^3}\right)}{x^3\cdot\left(-1-8\cdot\frac{1}{x^3}\right)}$$

$$= \lim_{x\to-\infty} \frac{2-4\cdot\frac{1}{x^2}+9\cdot\frac{1}{x^3}}{-1-8\cdot\frac{1}{x^3}} = \frac{2}{-1} = \boxed{-2},$$

since $\lim_{x\to-\infty} \frac{1}{x^2} = 0 = \lim_{x\to-\infty} \frac{1}{x^3}$.

2b) $$\lim_{x\to\infty} \left(2x-\sqrt{4x^2-3x+5}\right)$$

$$= \lim_{x\to\infty} \frac{4x^2-(4x^2-3x+5)}{2x+\sqrt{4x^2-3x+5}}$$

$$= \lim_{x\to\infty} \frac{3x-5}{2x+\sqrt{4x^2-3x+5}}$$

$$= \lim_{x\to\infty} \frac{3-5\cdot\frac{1}{x}}{2+\sqrt{4-3\cdot\frac{1}{x}+5\cdot\frac{1}{x^2}}} = \frac{3}{2+\sqrt{4}} = \boxed{\frac{3}{4}}$$

using again $\lim_{x\to\infty} \frac{1}{x} = 0 = \lim_{x\to\infty} \frac{1}{x^2}$.

3a) x is in $D(f) \Leftrightarrow x-4 \neq 0$ and $25-x^2 \geq 0$

$\Leftrightarrow x \neq 4$ and $|x| \leq 5$

Hence $D(f) = \boxed{[-5,+4) \cup (+4,+5]}$.

3b) We compute $\lim\limits_{x\to 4} \dfrac{3-\sqrt{25-x^2}}{x-4}$ $\left(= \text{"}\dfrac{0}{0}\text{"}\right)$

$$= \lim_{x\to 4} \frac{9-(25-x^2)}{(x-4)\cdot(3+\sqrt{25-x^2})}$$

$$= \lim_{x\to 4} \frac{x^2-16}{x-4} \cdot \lim_{x\to 4} \frac{1}{3+\sqrt{25-x^2}}$$

$$= \lim_{x\to 4} (x+4) \cdot \frac{1}{3+\sqrt{9}} = \frac{8}{6} = \frac{4}{3}$$

Since $\lim\limits_{x\to 4}$ exists, f has a removable singularity in $x=4$. The continuous extension is given by

$$\boxed{F(x) = \begin{cases} f(x) & , \text{ if } x \neq 4 \\ \frac{4}{3} & , \text{ if } x = 4. \end{cases}}$$

4) We use the <u>Intermediate Value Theorem</u>:
for the continuous function $f(x) = \cos x + \frac{x^2}{4\pi}$:
Since $f(0) = 1 > 0$ and $f(\pi) = -1 + \frac{\pi^2}{4\pi}$
$= -1 + \frac{\pi}{4} < 0$ since $\pi < 4$, there exists
some $x$ in $[0, 2\pi]$ with $f(x) = 0$.

5a) We compute $0 = \frac{d}{dx}\left(\cos y + \frac{y^2}{x}\right)$

$$= -\sin y \cdot \frac{dy}{dx} + \frac{2y \cdot \frac{dy}{dx} \cdot x - y^2}{x^2}$$

$$= \frac{dy}{dx}\left(\frac{2y}{x} - \sin y\right) - \frac{y^2}{x^2}$$ and hence

$$\frac{dy}{dx} = \boxed{\frac{y^2}{2yx - x^2 \cdot \sin y}}$$

5b) For $x = \pi^2$ and $y = \pi$ we get

$$\frac{dy}{dx} = \frac{\pi^2}{2 \cdot \pi \cdot \pi^2 - \pi^4 \cdot \sin \pi} = \frac{1}{2\pi}$$

The equation for the line through $P(\pi^2, \pi)$
with slope $m = \frac{1}{2\pi}$ is

$$\boxed{y = \frac{1}{2\pi}(x - \pi^2) + \pi \left(= \frac{1}{2\pi} \cdot x + \frac{\pi}{2}\right)}$$

6) We apply the Mean Value Thm to the differentiable function $\boxed{f(x) = \tan\left(x - \frac{x^2}{2\pi}\right)}$ on $\left[0, \frac{\pi}{2}\right)$ and set $a = 0$, $b = x$:

$$\frac{f(x) - f(0)}{x - 0} = \frac{\tan\left(x - \frac{x^2}{2\pi}\right)(-0)}{x} = f'(c)$$

for some $c$ in $(0, x)$. Computing the derivative we get

$$f'(c) = \frac{1}{\cos^2\left(c - \frac{c^2}{2\pi}\right)} \cdot \left(1 - \frac{2c}{2\pi}\right), \text{ where}$$

$$\cos^2\left(c - \frac{c^2}{2\pi}\right) \leq 1 \text{ and } 1 - \frac{2c}{2\pi} > 1 - \frac{1}{2} = \frac{1}{2},$$

so that $\boxed{f'(c) \geq \frac{1}{2}}$ for any $c$ in $(0, x)$.

Hence $\frac{\tan\left(x - \frac{x^2}{2\pi}\right)}{x} \geq \frac{1}{2}$, so that

$$\tan\left(x - \frac{x^2}{2\pi}\right) \geq \frac{x}{2} \text{ for all } x \text{ in } \left[0, \frac{\pi}{2}\right).$$