Resit Calculus 1, 07-01-2019, Solutions.

1. a) The expression under the square-root has to be non-negative, so we need:

$$8x - 3x^2 \geq 0 \Longrightarrow x(8-3x) \geq 0 \Longrightarrow 0 \leq x \leq \frac{8}{3}.$$

Therefore $D_f = [0, \frac{8}{3}]$.

b) There are two boundary points, $x = 0$ and $x = \frac{8}{3}$, where f clearly has an absolute minimum 0. Now calculate the derivative on $(0, \frac{8}{3})$:

$$f'(x) = \sqrt{8x - 3x^2} + \frac{x(8-6x)}{2\sqrt{8x-3x^2}} = \frac{12x - 6x^2}{\sqrt{8x-3x^2}},$$

so the only critical point is $x = 2$. Since $f'(x) > 0$ [so f is increasing] on $(0, 2)$ and $f'(x) < 0$ [so f is decreasing] on $(2, \frac{8}{3})$ the function has a absolute maximum $f(2) = 4$.

c) Calculate the second derivative:

$$\begin{aligned} f''(x) &= \frac{(12-12x)\sqrt{8x-3x^2} - \frac{(12x-6x^2)(8-6x)}{2\sqrt{8x-3x^2}}}{8x-3x^2} = \\ &= \frac{(12-12x)(8x-3x^2) - (6x-3x^2)(8-6x)}{(8x-3x^2)^{3/2}} = \frac{6x(3x^2-12x+8)}{(8x-3x^2)^{3/2}}. \end{aligned}$$

It is clear that for small (positive) values of x we have $f''(x) > 0$, so f can't be concave down on its domain.

2. a) We can use l'Hospitals rule twice for the whole fraction, but the calculations become somewhat easier if we split the limit in two parts and use l'Hospitals rule on both parts:

$$\lim_{x\to 1} \frac{\sin(\pi x)}{x-1} = \lim_{x\to 1} \frac{\pi\cos(\pi x)}{1} \overset{(H)}{=} -\pi$$

and

$$\lim_{x\to 1} \frac{\ln(x)}{x-1} \overset{(H)}{=} \lim_{x\to 1} \frac{\frac{1}{x}}{1} = 1.$$

So

$$\lim_{x\to 1} \frac{\sin(\pi x)\ln(x)}{(x-1)^2} = -\pi \cdot 1 = -\pi.$$

b) Multiply numerator and denominator by the conjugate of this expression:

$$\begin{aligned} &\lim_{x\to\infty} \left(x^2 - \sqrt{x^4 + 5x^2 + x\cos(x)}\right) \times \frac{x^2 + \sqrt{x^4 + 5x^2 + x\cos(x)}}{x^2 + \sqrt{x^4 + 5x^2 + x\cos(x)}} \\ &= \lim_{x\to\infty} \frac{-5x^2 - x\cos(x)}{x^2 + \sqrt{x^4 + 5x^2 + x\cos(x)}} \\ &= \lim_{x\to\infty} \frac{-5 - \frac{\cos(x)}{x}}{1 + \sqrt{1 + \frac{5}{x^2} + \frac{\cos(x)}{x^3}}} = \frac{-5}{1+1} = -\frac{5}{2}, \end{aligned}$$

since by the squeeze law $\lim_{x\to\infty} \frac{\cos(x)}{x} = 0$ and $\lim_{x\to\infty} \frac{\cos(x)}{x^3} = 0$.

3. a) For continuity we must have

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) = f(0) = b.$$

Now

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \sin\left(2x - \frac{\pi}{6}\right) + 1 = \sin\left(-\frac{\pi}{6}\right) + 1 = -\frac{1}{2} + 1 = \frac{1}{2},$$

so $b = \frac{1}{2}$ and a can be any real number.

b) First of all f has to be continuous at $x = 0$, so $b = \frac{1}{2}$. Then, for $x > 0$ we have $f'(x) = 2\cos\left(2x - \frac{\pi}{6}\right)$, and therefore $f'_+(0) = \lim_{x\to 0+} f'(x) = 2\cos\left(-\frac{\pi}{6}\right) = \sqrt{3}$. And for $x < 0$ we have $f'(x) = a$, so also $f'_-(x) = \lim_{x\to 0-} f'(x) = a$. Therefore f is differentiable at $x = 0$ if $a = \sqrt{3}$ and $b = \frac{1}{2}$.

4. a) Use the fact that $x^{\sqrt{x}} = e^{\sqrt{x}\ln(x)}$. Then calculate the derivative:

$$f'(x) = e^{\sqrt{x}\ln(x)}\left(\frac{\ln(x)}{2\sqrt{x}} + \frac{\sqrt{x}}{x}\right) = x^{\sqrt{x}}\left(\frac{2+\ln(x)}{2\sqrt{x}}\right) \quad \text{for all } x > 0.$$

b) Since $f(1) = 1$ and $f'(1) = 1$ we get $L(x) = f(1) + f'(1)(x-1) = x$.

c) Since $f'(x) < 0$ [so f is decreasing] on $(0, e^{-2})$ and $f'(x) > 0$ [so f is increasing] on (e^{-2}, ∞), the continuous function f is not one-to-one on $[0, \infty)$.

5. Let $f(x) = \sqrt[5]{x}$. Then f is continuous on $[32, 33]$ and differentiable on $(32, 33)$. So according to the Mean Value Theorem there exists a c in $(32, 33)$ such that:

$$\frac{\sqrt[5]{33} - \sqrt[5]{32}}{33 - 32} = f'(c) = \frac{1}{5\sqrt[5]{c^4}}.$$

We have $\sqrt[5]{32} = 2$ and since $c > 32$, so $\sqrt[5]{c^4} > 2^4 = 16$, we also have

$$\frac{1}{5\sqrt[5]{c^4}} < \frac{1}{80} = 0.0125.$$

This yields

$$2 = \sqrt[5]{32} < \sqrt[5]{33} = 2 + \frac{1}{5\sqrt[5]{c^4}} < 2 + 0.0125 = 2.0125.$$

6. Since $f(x) = \arctan(\sqrt{x})$ we have

$$\begin{cases} f(1) = \arctan(1) = \frac{\pi}{4}, & \\ f'(x) = \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}}, & \text{so } f'(1) = \frac{1}{4}, \\ f''(x) = -\frac{1+3x}{4x\sqrt{x}(1+x)^2}, & \text{so } f''(1) = -\frac{1}{4}. \end{cases}$$

Therefore

$$P_2(x) = \frac{\pi}{4} + \frac{1}{4}(x-1) - \frac{1}{8}(x-1)^2.$$

7. a) Use for example the substitution $t = \ln(\cos(x))$, so that $dt = \frac{1}{\cos(x)} \cdot -\sin(x)\,dx = -\tan(x)\,dx$:

$$\int \tan(x)\ln(\cos(x))\,dx = -\int t\,dt = -\frac{1}{2}t^2 + C = -\frac{1}{2}\ln^2(\cos(x)) + C.$$

b) This is an improper integral of the first kind. We first derive an anti-derivative with the method of partial fraction decomposition:

$$\int \frac{6}{x^2+3x}\,dx = \int \frac{6}{x(x+3)}\,dx = \int \frac{2}{x} - \frac{2}{x+3}\,dx = 2\ln|x| - 2\ln|x+3| + C.$$

Then

$$\int_1^\infty \frac{6}{x^2+3x}\,dx = \lim_{t\to\infty}\int_1^t \frac{6}{x^2+3x}\,dx = \lim_{t\to\infty}\left[2\ln(x) - 2\ln(x+3)\Big|_1^t\right]$$

$$= \lim_{t\to\infty} 2\ln\left(\frac{t}{t+3}\right) + 2\ln(4) = 2\ln(4) = 4\ln(2).$$

c) Use integration by parts and long-division:

$$\int_0^1 x\ln(x+1)\,dx = \frac{1}{2}x^2\ln(x+1)\Big|_0^1 - \int_0^1 \frac{1}{2}\frac{x^2}{x+1}\,dx =$$

$$= \frac{1}{2}\ln(2) - \frac{1}{2}\int_0^1 x - 1 + \frac{1}{x+1}\,dx = \frac{1}{2}\ln(2) - \frac{1}{2}\left(\frac{1}{2}x^2 - x + \ln(x+1)\Big|_0^1\right) = \frac{1}{4}.$$

8. This is an improper integral of the first and second kind, but we cannot find an antiderivative easily. So we split the integral in two parts and use a comparison test for each part.
Part 1: Since on $(0,1)$

$$0 < \frac{2-\sin(x^2)}{(1+x)\sqrt{x}} < \frac{2}{(1+x)\sqrt{x}} < \frac{2}{\sqrt{x}}$$

and since

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx \text{ is convergent } (p\text{-integral with } p = \tfrac{1}{2}),$$

the given integral is also convergent on $(0,1)$.
Part 2: Since on $(1,\infty)$

$$0 < \frac{2-\sin(x^2)}{(1+x)\sqrt{x}} < \frac{3}{(1+x)\sqrt{x}} < \frac{3}{x\sqrt{x}}$$

and since

$$\int_0^1 \frac{1}{x\sqrt{x}}\,dx \text{ is convergent } (p\text{-integral with } p = \tfrac{3}{2}),$$

the given integral is also convergent on $(1,\infty)$.
The conclusion is that the given integral is convergent.