Second test Calculus 1, 22-10-2018, Solutions.

1. a) Use the chain-rule to find that

$$f'(x) = \frac{1}{1+\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} > 0 \text{ for all } x > 0,$$

so f is increasing on $[0, \infty)$ and therefore one-to-one.

b) First remark that f is continuous and increasing, $f(0) = 0$ and $\lim_{x\to\infty} f(x) = \infty$, so that the range of f is $[0, \infty)$. The domain of f^{-1} is equal to the range of f, so the domain of f^{-1} is $[0, \infty)$. Now let $y = f^{-1}(x)$. Then

$$x = f(y) = \ln(1 + \sqrt{y}) \Longrightarrow e^x = 1 + \sqrt{y} \Longrightarrow y = f^{-1}(x) = (e^x - 1)^2 .$$

2. a) Since $\lim_{x\to 0+} \sqrt{x}\ln(x) = 0$ (standard limit: "x^a wins over $\ln x$ for all $a > 0$") we also have $\lim_{x\to 0+} x(\ln(x))^2 = \lim_{x\to 0+} \left(\sqrt{x}\ln(x)\right)^2 = 0$.

b) There are no singular points, so we only have to consider critical points, the boundary point $x = e$ and the behavior of f when x tends to $0+$ (which is already done in exercise 2a). First calculate the derivative:

$$f'(x) = (\ln(x))^2 + 2\ln(x) = \ln(x)\,(\ln(x) + 2)\,.$$

Now $f'(x) = 0$ implies $\ln(x) = 0$, so $x = 1$, or $\ln(x) = -2$, so $x = e^{-2}$. Now $f'(x) > 0$ (so f is increasing) on $(0, e^{-2})$ and on $(1, e)$ and $f'(x) < 0$ (so f is decreasing) on $(e^{-2}, 1)$. We find $f(1) = 0$, $f(e^{-2}) = 4e^{-2}$ and $f(e) = e$. Since $\lim_{x\to 0+} \sqrt{x}\ln(x) = 0$, we can conclude: f has an absolute maximum e (for $x = e$), a local maximum $4e^{-2}$ for $x = e^{-2}$, and an absolute minimum 0 for $x = 1$.

c) Calculate

$$f''(x) = \frac{2\ln(x)}{x} + \frac{2}{x} = \frac{2(\ln(x) + 1)}{x},$$

so $f''(x) < 0$ only on $(0, e^{-1})$ and therefore f is concave down on $(0, e^{-1})$.

3. Rewrite the limit to one fraction and remark that we get a $0/0$ situation. Then we use l'Hospital's rule twice:

$$\lim_{x\to 0}\left(\frac{1}{e^x - 1} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x + 1 - e^x}{x(e^x - 1)} =$$

$$\overset{(H)}{=} \lim_{x\to 0} \frac{1 - e^x}{xe^x + e^x - 1} \overset{(H)}{=} \lim_{x\to 0} \frac{-e^x}{xe^x + 2e^x} = -\frac{1}{2}.$$

4. a) Calculate $f(0) = 1$ and $f'(x) = \frac{-2}{(1+2x)^2}$ so $f'(0) = -2$. Therefore $L(x) = f(0) + f'(0)(x - 0) = 1 - 2x$.

b) $L(0.005) = 1 - 2 \cdot 0.005 = 0.99$ which is the linear approximation of $\frac{1}{1.01}$. Since $f''(x) = \frac{8}{(1+2x)^3}$ the error-function is

$$E(x) = \frac{f''(s)(x - 0)^2}{2} = \frac{4x^2}{(1 + 2s)^3}, \qquad \text{with } 0 < s < x.$$

So for the absolute value of the error we have:

$$|E(0.005)| = \left|\frac{4(0.005)^2}{(1+2s)^3}\right| < 4(0.005)^2 = 10^{-4}.$$

5. With the Fundamental Theorem of Calculus and the chain-rule we find:

$$f'(x) = \arctan\left(4\sin^2(x)\right)\cdot\cos(x).$$

So (use $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $\cos\left(\frac{\pi}{6}\right) = \frac{1}{2}\sqrt{3}$):

$$f'\left(\tfrac{\pi}{6}\right) = \arctan\left(4\sin^2\left(\tfrac{\pi}{6}\right)\right)\cdot\cos\left(\tfrac{\pi}{6}\right) = \arctan(1)\cdot\frac{1}{2}\sqrt{3} = \frac{1}{8}\pi\sqrt{3}.$$

6. a) Use the trigonometric formula $\cos(2t) = 1-2\sin^2(t)$, so $\sin^2(t) = \frac{1}{2}-\frac{1}{2}\cos(2t)$:

$$\int \sin^2(3x)\,dx = \int \frac{1}{2} - \frac{1}{2}\cos(6x)\,dx$$
$$= \frac{1}{2}x - \frac{1}{12}\sin(6x) + C.$$

b) We start with calculating an antiderivative. Use integration by parts to find

$$\int \frac{\ln x}{x^3}\,dx = -\frac{\ln(x)}{2x^2} + \int \frac{1}{2x^3}\,dx = -\frac{\ln(x)}{2x^2} - \frac{1}{4x^2}.$$

Then we calculate the improper integral:

$$\int_1^\infty \frac{\ln x}{x^3}\,dx = \lim_{R\to\infty}\int_1^R \frac{\ln x}{x^3}\,dx = \lim_{R\to\infty}\left(-\frac{\ln(R)}{2R^2} - \frac{1}{4R^2}\right) - \left(0 - \frac{1}{4}\right) = \frac{1}{4},$$

since "R^2 wins over $\ln(R)$ if $R\to\infty$".

c) Use partial fraction decomposition (taking into account the double linear factor x^2!):

$$\int_1^2 \frac{3x+2}{x^2(x+1)}\,dx = \int_1^2 \frac{1}{x} + \frac{2}{x^2} - \frac{1}{x+1}\,dx$$
$$= \ln|x| - \frac{2}{x} - \ln|x+1|\Big|_1^2 = 1 + 2\ln(2) - \ln(3).$$

7. This is an improper integral of the first kind. Since $\lim\limits_{x\to\infty}\arctan(x^2) = \frac{\pi}{2} < 2$ we have $\dfrac{1}{x\arctan(x^2)} > \dfrac{1}{2x}$, for large values of x. And (p-integral with $p=1$)

$$\int_1^\infty \frac{1}{2x}\,dx = \frac{1}{2}\int_1^\infty \frac{1}{x}\,dx \text{ is divergent to } \infty.$$

So $\int\limits_1^\infty \dfrac{1}{x\arctan(x^2)}\,dx$ is divergent to ∞ as well.