Resit Calculus 1, 05-02-2018, Solutions.

1. Move every term to the left hand side of the inequality sign and make one fraction:

$$\frac{x-1}{x+2} - 3 > 0 \Longrightarrow \frac{x-1-3(x+2)}{x+2} > 0$$
$$\Longrightarrow \frac{-2x-7}{x+2} > 0 \Longrightarrow \frac{2x+7}{x+2} < 0.$$

The left-hand side can only be negative if one factor is positive and the other is negative. The first option, $2x+7>0$ and $x+2<0$, yields $-\frac{7}{2} < x < -2$. The other option, $2x+7<0$ and $x+2>0$, has no solution. So the solution set is $S = (-\frac{7}{2}, -2)$. [Of course you can also organize this sign information in a chart, as is presented in Adams, section P.1.]

2. a) $\lim_{x\to 0+} \frac{\ln(x)}{\sqrt{x}} = -\infty$ (the limit is of the form $-\infty$ divided by $0+$)
   and $\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} = 0$ (standard limit: "$\sqrt{x}$ wins over $\ln(x)$")

   b) There are no boundary points and no singular points, so we only have to consider critical points, and the behavior of f when x tends to $0+$ or ∞ (which is already done in exercise 2a). First calculate the derivative:

$$f'(x) = \frac{\frac{1}{x}\cdot\sqrt{x} - \frac{1}{2\sqrt{x}}\cdot\ln(x)}{x} = \frac{2-\ln(x)}{2x\sqrt{x}}.$$

   Now $f'(x) = 0$ implies $\ln(x) = 2$, so $x = e^2$. Since $f'(x) > 0$ (so f is increasing) on $(0, e^2)$ and since $f'(x) < 0$ (so f is decreasing) on (e^2, ∞), f has an absolute maximum in $x = e^2$ with value $f(e^2) = \frac{2}{e}$. There is no minimum value!

   c) Calculate

$$f''(x) = \frac{-\frac{1}{x}\cdot 2x\sqrt{x} - 3\sqrt{x}\cdot(2-\ln(x))}{4x^3} = \frac{3\ln(x) - 8}{4x^2\sqrt{x}},$$

   so $f''(x) = 0$ implies that $\ln(x) = \frac{8}{3}$, thus $x = e^{8/3}$. Since $f''(x) < 0$ on $(0, e^{8/3})$ and since $f''(x) > 0$ on $(e^{8/3}, \infty)$, so $f''(x)$ changes sign at $x = e^{8/3}$, the curve $y = f(x)$ has an inflection point $(e^{8/3}, \frac{8}{3}e^{-4/3})$.

3. a) In the neighborhood of $x = 0$ we can write $|x^2 - 1| = 1 - x^2$. So the limit becomes:

$$\lim_{x\to 0} \frac{(1-x^2)-1}{\sin(x)} = \lim_{x\to 0} \frac{-x^2}{\sin(x)} = \lim_{x\to 0} -x\cdot\frac{x}{\sin(x)} = 0\cdot 1 = 0.$$

   b) Use $a^b = e^{b\ln(a)}$: $\lim_{x\to 0} \Big(1+\sin(2x)\Big)^{3/x} = \lim_{x\to 0} e^{\left(\frac{3\ln(1+\sin(2x))}{x}\right)} = e^6,$
   since l'Hospitals rule ($\left[\frac{0}{0}\right]$-situation) gives

$$\lim_{x\to 0} \frac{3\ln(1+\sin(2x))}{x} = \lim_{x\to 0} \frac{3\cdot 2\cos(2x)}{1+\sin(2x)} = 6.$$

4. If $f'(0)$ exists it must be equal to:

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{h+2h^2\sin\left(\frac{1}{h}\right)-0}{h}=\lim_{h\to 0}1+2h\sin\left(\frac{1}{h}\right).$$

The last limit equals 0 and can be calculated using the squeeze theorem. Since

$$1-2|h|\leq 1+2h\sin\left(\frac{1}{h}\right)\leq 1+2|h|$$

and since $\pm|h|$ tend to 0 if h tends to 0, we find $f'(0)=1$.

5. a) Calculate the derivative: $f'(x)=\frac{1}{\sqrt{x}}+\frac{1}{x}>0$ for all x. So f is strictly increasing on $(0,\infty)$ and therefore one-to-one on $(0,\infty)$.

   b) Since f is one-to-one and differentiable, there exists a differentiable inverse-function g. Remark that $f(1)=2$, so that $1=g(2)$. This yields

$$g'(2)=\frac{1}{f'(g(2))}=\frac{1}{f'(1)}=\frac{1}{1+1}=\frac{1}{2}.$$

6. Choose $x>0$ arbitrarily and define the function $f(t)=\sqrt[3]{8+5t}=(8+5t)^{1/3}$. Then f is continuous on $[0,x]$ and differentiable on $(0,x)$. So according to the Mean Value Theorem there exists a c in $(0,x)$ such that:

$$\frac{\sqrt[3]{8+5x}-2}{x}=\frac{f(x)-f(0)}{x-0}=f'(c)=\frac{5}{3}(8+5c)^{-2/3}<\frac{5}{3}\cdot\frac{1}{4}=\frac{5}{12},$$

since $c>0$, so $8+5c>8$ and thus $(8+5c)^{2/3}>8^{2/3}=4$.
Now multiply both sides by x and shift 2 to the right-hand side, to find that $\sqrt[3]{8+5x}<2+\frac{5}{12}x$ for all $x>0$.

7. Introduce $f(x)=x\sin(\pi x)$ and then calculate

$$\begin{cases} f(-1)=-\sin(-\pi)=0, & \\ f'(x)=\sin(\pi x)+x\pi\cos(\pi x), & \text{so } f'(-1)=\pi, \\ f''(x)=2\pi\cos(\pi x)-x\pi^2\sin(\pi x), & \text{so } f''(-1)=-2\pi. \end{cases}$$

Therefore

$$P_2(x)=\pi(x+1)-\pi(x+1)^2.$$

8. a) First use long division and factorize the denominator:

$$\int_3^6\frac{3x^2}{x^2-x-2}\,dx=\int_3^6 3+\frac{3x+6}{(x-2)(x+1)}\,dx.$$

   Now use the method of partial fraction decomposition:

$$\frac{3x+6}{(x-2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+1}=\frac{A(x+1)+B(x-2)}{(x-2)(x+1)}.$$

Equate the coefficients of x and the constant term to obtain $A = 4$ and $B = -1$. So we find

$$\int_3^6 3 + \frac{3x+6}{(x-2)(x+1)}\,dx = \int_3^6 3 + \frac{4}{x-2} - \frac{1}{x+1}\,dt$$

$$= 3x + 4\ln|x-2| - \ln|x+1|\Big|_3^6 = 9 + 5\ln(4) - \ln(7).$$

b) Use the substitution $t = \sqrt{x}$ (so $dt = \frac{1}{2\sqrt{x}}\,dx$ and thus $dx = 2\sqrt{x}\,dt = 2t\,dt$) followed by integration by parts:

$$\int_0^1 \arctan(\sqrt{x})\,dx = \int_0^1 2t\arctan(t)\,dt = t^2\arctan(t)\Big|_0^1 - \int_0^1 \frac{t^2}{1+t^2}\,dt$$

$$= \frac{\pi}{4} - \int_0^1 1 - \frac{1}{1+t^2}\,dt = \frac{\pi}{4} - \left(t - \arctan(t)\Big|_0^1\right) = \frac{\pi}{2} - 1.$$

9. This is an improper integral of the second kind, but we cannot find an antiderivative easily. So we use a comparison test. Since on $(0,\pi)$

$$0 < \frac{3+\cos(x)}{(1+x^2)\sqrt{x}} < \frac{4}{(1+x^2)\sqrt{x}} < \frac{4}{\sqrt{x}}$$

and since

$$\int_0^\pi \frac{4}{\sqrt{x}}\,dx \text{ is convergent } (p\text{-integral with } p = \tfrac{1}{2}),$$

the given integral is also convergent.