Second test Calculus 1, 23-10-2017, Solutions.

1. a) $\lim_{x\to 0+} x^2 - 2x^2\ln x = 0 - 0 = 0$ (the second part is a standard limit: "x^2 wins over $\ln x$") and $\lim_{x\to\infty} x^2 - 2x^2\ln x = \lim_{x\to\infty} x^2(1-2\ln x) = \infty\cdot(-\infty) = -\infty$.

   b) There are no boundary points and no singular points, so we only have to consider critical points, and the behavior of f when x tends to $0+$ or ∞ (which is already done in exercise 1a). First calculate the derivative $f'(x) = 2x - 4x\ln x - 2x = -4x\ln x$. Now $f'(x) = 0$ implies $\ln x = 0$, so $x = 1$. Since $f'(x) > 0$ (so f is increasing) on $(0,1)$ and since $f'(x) < 0$ (so f is decreasing) on $(1,\infty)$, f has an absolute maximum in $x = 1$ with value $f(1) = 1$. There is no minimum value!

   c) Calculate $f''(x) = -4\ln x - 4$, so $f''(x) = 0$ implies that $\ln x = -1$, thus $x = \frac{1}{e}$. Since $f''(x) > 0$ on $(0,\frac{1}{e})$ and since $f''(x) < 0$ on $(\frac{1}{e},\infty)$, so $f''(x)$ changes sign at $x = \frac{1}{e}$, the curve $y = f(x)$ has an inflection point $(\frac{1}{e},\frac{3}{e^2})$.

2. a) First remark that the denominator is always positive so that the domain of f is $\mathbb{R}$. Since f is continuous and $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$ the range of f is also $\mathbb{R}$. Now use the quotient-rule to find that

$$f'(x) = \frac{5x^4(x^2+1) - 2x\cdot x^5}{(x^2+1)^2} = \frac{x^4(3x^2+5)}{(x^2+1)^2} > 0 \text{ for all } x \in \mathbb{R}\setminus\{0\},$$

   so f is increasing on $\mathbb{R}$ and therefore one-to-one. So the inverse-function f^{-1} exists. The domain of f^{-1} is equal to the range of f, so the domain of f^{-1} is $\mathbb{R}$.

   b) Remark that $f(1) = \frac{1}{2}$, so that $f^{-1}(\frac{1}{2}) = 1$. This yields

$$(f^{-1})'\left(\frac{1}{2}\right) = \frac{1}{f'(f^{-1}(\frac{1}{2}))} = \frac{1}{f'(1)} = \left(\frac{3+5}{(1+1)^2}\right)^{-1} = \frac{1}{2}.$$

3. This is an $\infty\cdot 0$ situation, so we have to rewrite the limit to create a $0/0$ situation. Then we use l'Hospital's rule:

$$\begin{aligned}\lim_{x\to\infty} x\,(\pi - 2\arctan x) &= \lim_{x\to\infty}\frac{\pi - 2\arctan x}{\frac{1}{x}} \overset{(H)}{=} \lim_{x\to\infty}\frac{\frac{-2}{1+x^2}}{-\frac{1}{x^2}} \\ &= \lim_{x\to\infty}\frac{2x^2}{1+x^2} = \lim_{x\to\infty}\frac{2}{\frac{1}{x^2}+1} = 2.\end{aligned}$$

4. a) $f(100) = \sqrt{100} = 10$ and $f'(x) = \frac{1}{2\sqrt{x}}$ so $f'(100) = \frac{1}{20}$. Therefore

$$L(x) = f(100) + f'(100)(x-100) = 10 + \frac{1}{20}(x-100) = 5 + \frac{1}{20}x.$$

b) $L(102) = 10 + \frac{1}{20}(102 - 100) = 10.1$ which is the linear approximation of $\sqrt{102}$. Since $f''(x) = \frac{-1}{4x\sqrt{x}}$ the error-function is

$$E(x) = \frac{f''(s)(x-100)^2}{2} = -\frac{(x-100)^2}{8s\sqrt{s}}, \qquad \text{with } 100 < s < 102,$$

and therefore $|\frac{1}{s\sqrt{s}}| < \frac{1}{1000}$. So for the absolute value of the error we have:

$$|E(102)| = \left| -\frac{(102-100)^2}{8s\sqrt{s}} \right| = \frac{1}{2s\sqrt{s}} < \frac{1}{2000} = 0.0005.$$

5. Use the Maclaurin polynomial of order $2n+1$ for e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + O\left(x^{2n+2}\right)$$

so after replacing x with $-x$:

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots - \frac{x^{2n+1}}{(2n+1)!} + O\left(x^{2n+2}\right).$$

After subtracting these expressions and dividing by 2, we get:

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + O\left(x^{2n+2}\right) = P_{2n+1}(x) + O\left(x^{2n+2}\right),$$

where

$$P_{2n+1}(x) = x + \frac{x^3}{3!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} = \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!}$$

is the requested Maclaurin polynomial.

6. a) Use integration by parts:

$$\int x \arctan x \, dx = \frac{1}{2}x^2 \arctan x - \int \frac{\frac{1}{2}x^2}{1+x^2} \, dx$$

$$= \frac{1}{2}x^2 \arctan x - \frac{1}{2}\int 1 - \frac{1}{1+x^2} \, dx = \frac{1}{2}x^2 \arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x + C.$$

b) Use the substitution $t = \ln x$ (so $dt = \frac{1}{x}\,dx$ and t goes from $\ln 1 = 0$ to $\ln \sqrt{e} = \frac{1}{2}$):

$$\int_1^{\sqrt{e}} \frac{\sin(\pi \ln x)}{x} \, dx = \int_0^{1/2} \sin(\pi t) \, dt = \left[-\frac{1}{\pi}\cos(\pi t) \right]_0^{1/2} = 0 - (-\frac{1}{\pi}) = \frac{1}{\pi}.$$

c) We start with calculating an antiderivative. Factorize the denominator and use partial fraction decomposition:

$$\int \frac{1}{x^2-4} \, dx = \int \frac{1}{(x+2)(x-2)} \, dx$$

$$= \int \frac{1/4}{x-2} - \frac{1/4}{x+2} \, dx = \frac{1}{4}\left(\ln|x-2| - \ln|x+2|\right) = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|.$$

Then we calculate the improper integral:

$$\int_3^{\infty} \frac{1}{x^2-4}\,dx = \lim_{t\to\infty}\left[\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|\right]_3^t = \lim_{t\to\infty}\frac{1}{4}\ln\left|\frac{t-2}{t+2}\right| - \frac{1}{4}\ln\left(\frac{1}{5}\right) = \frac{1}{4}\ln 5.$$

7. This is an improper integral of the first and second kind. So split into two parts (I_1 and I_2) and consider each part separately:

   (i) On $[0,1]$: $I_1 = \int\limits_0^1 \frac{1}{(1+x^3)\sqrt{x}}\,dx$. Since $\frac{1}{(1+x^3)\sqrt{x}} < \frac{1}{\sqrt{x}}$ and since $\int\limits_0^1 \frac{1}{\sqrt{x}}\,dx$ is convergent (p-integral with $p = \frac{1}{2} < 1$), I_1 is also convergent.

   (ii) On $[1,\infty)$: $I_2 = \int\limits_0^{\infty} \frac{1}{(1+x^3)\sqrt{x}}\,dx$. Since $\frac{1}{(1+x^3)\sqrt{x}} < \frac{1}{x^3\sqrt{x}}$ and since $\int\limits_1^{\infty} \frac{1}{x^3\sqrt{x}}\,dx$ is convergent (p-integral with $p = \frac{7}{2} > 1$), I_2 is also convergent.

   Combining (i) and (ii) we conclude that the given improper integral is convergent.