First test Calculus 1, 25-09-2017, Solutions.

1. Move every term to the left hand side of the inequality sign and make one fraction:

$$\frac{1}{x+1} - 1 - \frac{x}{2} \leq 0 \Longrightarrow \frac{2 - 2(x+1) - x(x+1)}{2(x+1)} \leq 0$$

$$\Longrightarrow \frac{-x^2 - 3x}{2(x+1)} \leq 0 \Longrightarrow \frac{x(x+3)}{2(x+1)} \geq 0.$$

The left-hand side is 0 for $x = -3$ or $x = 0$. Furthermore it can only be positive if all three factors are positive (which is true for $x > 0$), or if one factor is positive and the other two are negative (which is true for $-3 < x < -1$). So the solution set is $S = [-3, -1) \cup [0, \infty)$. [Of course you can also organize this sign information in a chart, as is presented in Adams, section P.1.]

2. To prove this identity we will mainly use the double-angle formula

$$\cos(2t) = \cos^2(t) - \sin^2(t) = 1 - 2\sin^2(t) = 2\cos^2(t) - 1.$$

[Here we also used the identity $\sin^2(t) + \cos^2(t) = 1$.] Rewriting these formulas gives

$$1 + \cos(2t) = 2\cos^2(t) \qquad \text{and} \qquad 1 - \cos(2t) = 2\sin^2(t).$$

So we find

$$\frac{1 - \cos(x)}{1 + \cos(x)} = \frac{2\sin^2(x/2)}{2\cos^2(x/2)} = \tan^2\left(\frac{x}{2}\right).$$

3. a) Since $x^2 + 1 \geq 1$, the square-root is defined for every $x \in \mathbb{R}$. So we only have to exclude x that make the denominator zero. Solving $1 - \sqrt{1 + x^2} = 0$ gives $x = 0$. So $D_f = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$.

   b) Since for all $x \in D_f$ we have

$$f(-x) = \frac{-x}{1 - \sqrt{(-x)^2 + 1}} = -\frac{x}{1 - \sqrt{x^2 + 1}} = -f(x),$$

   we can conclude that f is an odd function (and not an even function).

   c) Calculate the derivative of f on D_f:

$$f'(x) = \frac{1 - \sqrt{x^2+1} - \frac{-2x^2}{2\sqrt{x^2+1}}}{(1 - \sqrt{x^2+1})^2} = \frac{(1 - \sqrt{x^2+1})\sqrt{x^2+1} + x^2}{(1 - \sqrt{x^2+1})^2\sqrt{x^2+1}} =$$

$$= \frac{\sqrt{x^2+1} - 1}{(1 - \sqrt{x^2+1})^2\sqrt{x^2+1}} = \frac{1}{(\sqrt{x^2+1} - 1)\sqrt{x^2+1}}.$$

   The derivative is always positive on D_f, so f is increasing on $(-\infty, 0)$ and on $(0, \infty)$ [and nowhere decreasing].

d) First the limits to ∞ and $-\infty$. Dividing both numerator and denominator by x gives:

$$\lim_{x\to\infty} \frac{x}{1-\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{1}{\frac{1}{x}-\sqrt{1+\frac{1}{x^2}}} = \frac{1}{0-1} = -1.$$

Now since f is an odd function it follows that $\lim_{x\to-\infty} f(x) = 1$ [or just do a similar calculation].

For the third limit we distinguish between $\lim_{x\to 0+}$ and $\lim_{x\to 0-}$. Multiply numerator and denominator by the conjugate of the denominator:

$$\lim_{x\to 0+} \frac{x}{1-\sqrt{x^2+1}} \times \frac{1+\sqrt{x^2+1}}{1+\sqrt{x^2+1}} = \lim_{x\to 0+} \frac{x(1+\sqrt{x^2+1})}{1-(x^2+1)}$$

$$= \lim_{x\to 0+} \frac{1+\sqrt{x^2+1}}{-x} = \frac{2}{0-} = -\infty.$$

Since f is an odd function it follows that $\lim_{x\to 0-} f(x) = +\infty$ [or just do a similar calculation]. The conclusion is that $\lim_{x\to 0} f(x)$ does not exist.

e) Combining the results of parts c) and d) the range R_f of f must be $(-\infty,-1)\cup(1,\infty)$.

4. a) You can also calculate the limit without the use of the double-angle formula. In that case make use of the fact that $\lim_{x\to 0} \frac{\sin(2x)}{2x} = 1$. Otherwise

$$\lim_{x\to 0} \frac{\sin(2x)}{x^2+3x} = \lim_{x\to 0} \frac{2\sin(x)\cos(x)}{x(x+3)} = \lim_{x\to 0} 2\cdot\frac{\sin(x)}{x}\cdot\frac{\cos(x)}{x+3} = 2\cdot 1\cdot\frac{1}{3} = \frac{2}{3}.$$

b) Use the fact that for $x > 2$ we have $|2x - x^2| = x^2 - 2x$. Then we find

$$\lim_{x\to 2+} \frac{|2x-x^2|}{4-x^2} = \lim_{x\to 2+} \frac{x^2-2x}{4-x^2} = \lim_{x\to 2+} \frac{x(x-2)}{(2-x)(2+x)} = \lim_{x\to 2+} \frac{-x}{2+x} = -\frac{1}{2}.$$

5. a) For continuity we must have

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) = f(0) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}.$$

Now

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} ax + b = b$$

while

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \tan\left(x+\frac{\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}.$$

So $b = \sqrt{3}$ and a can be any real number.

b) First of all f has to be continuous at $x = 0$, so $b = \sqrt{3}$. Then, for $x > 0$ we have

$$f'(x) = \frac{1}{\cos^2\left(x + \frac{\pi}{3}\right)}, \quad \text{and therefore } f'_+(0) = \lim_{x \to 0+} f'(x) = \frac{1}{\cos^2\left(\frac{\pi}{3}\right)} = 4.$$

And for $x < 0$ we have $f'(x) = a$, so also $f'_-(x) = a$. Therefore f is differentiable at $x = 0$ if $a = 4$ and $b = \sqrt{3}$.

6. a) We use implicit differentiation, the product rule and the chain rule to find:

$$3x^2 - 3y - 3x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0. \qquad (*)$$

Therefore:

$$(3y^2 - 3x)\frac{dy}{dx} = 3y - 3x^2, \qquad \text{which leads to} \qquad \frac{dy}{dx} = \frac{y - x^2}{y^2 - x}.$$

We continue with this result to find

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{y - x^2}{y^2 - x}\right) = \frac{(\frac{dy}{dx} - 2x)(y^2 - x) - (2y\frac{dy}{dx} - 1)(y - x^2)}{(y^2 - x)^2} \\ &= \frac{\left(\frac{y - x^2}{y^2 - x} - 2x\right)(y^2 - x) - \left(2y \cdot \frac{y - x^2}{y^2 - x} - 1\right)(y - x^2)}{(y^2 - x)^2}, \end{aligned}$$

which may be simplified a little bit more (but this is not necessary).
[You can also continue with implicit differentiation of formula $(*)$ and then substitute $\frac{dy}{dx}$ to find the same result.]

b) The slope of the tangent line is

$$\left.\frac{dy}{dx}\right|_{(1,0)} = \left.\frac{y - x^2}{y^2 - x}\right|_{(1,0)} = 1.$$

And therefore the equation of the tangent line is $y = 1(x - 1) + 0 = x - 1$.