Resit Calculus 1, 10-01-2017, Solutions.

1. a) The domain of f consists of all x for which $x \geq 0$ and $4\sqrt{x} - x \geq 0$. The second inequality gives $\sqrt{x} \leq 4$, so $x \leq 16$. So $D_f = [0, 16]$.

   b) On $[1, 9]$ we have

$$f'(x) = \frac{1}{2\sqrt{4\sqrt{x} - x}} \cdot \left(\frac{2}{\sqrt{x}} - 1 \right) = 0 \Longleftrightarrow x = 4.$$

   Since $f'(x) > 0$ (so f is strictly increasing) on $[1, 4)$ and since $f'(x) < 0$ (so f is strictly decreasing) on $(4, 9]$, f has an absolute maximum in $x = 4$ with value $f(4) = 2$. The endpoints of the interval give an absolute minimum with value $f(1) = f(9) = \sqrt{3}$.

2. a) Multiply numerator and denominator by the expression $\sqrt{1 + \sin x} + \sqrt{1 - \sin x}$ to get

$$\lim_{x \to 0} \frac{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}{x} \cdot \frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}} =$$

$$\lim_{x \to 0} 2 \cdot \frac{\sin x}{x} \cdot \frac{1}{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}} = 2 \cdot 1 \cdot \frac{1}{2} = 1,$$

   since $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$ (standard limit, or use l'Hospital).

   b) Use $a^b = e^{b \ln (a)}$:

$$\lim_{x \to 0} \Big(1 + \tan (2x) \Big)^{1/x} = \lim_{x \to 0} e^{\dfrac{\ln (1 + \tan (2x))}{x}} = e^2,$$

   since l'Hospital ($\left[\frac{0}{0}\right]$-situation) gives

$$\lim_{x \to 0} \frac{\ln (1 + \tan (2x))}{x} = \lim_{x \to 0} \frac{1}{1 + \tan (2x)} \cdot \frac{2}{\cos^2 (2x)} = 2.$$

3. Function f is differentiable in 0 if $f'(0) = \lim\limits_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ exists. Here we get (use l'Hospital twice):

$$f'(0) = \lim_{x \to 0} \frac{\frac{e^{2x} - 1}{x} - 2}{x - 0} = \lim_{x \to 0} \frac{e^{2x} - 1 - 2x}{x^2} \overset{(H)}{=} \lim_{x \to 0} \frac{2e^{2x} - 2}{2x} \overset{(H)}{=} \lim_{x \to 0} \frac{4e^{2x}}{2} = 2.$$

   So f is differentiable in 0 and $f'(0) = 2$.

4. Choose $x > 0$ arbitrarily and define the function $f(t) = (1 + t)^{3/2}$. Then f is continuous on $[0, x]$ and differentiable on $(0, x)$. So according to the Mean Value Theorem there exists a c in $(0, x)$ such that:

$$\frac{(1 + x)^{3/2} - 1}{x} = \frac{f(x) - f(0)}{x - 0} = f'(c) = \frac{3}{2}(1 + c)^{1/2} > \frac{3}{2},$$

   since $c > 0$, so $1 + c > 1$ and thus $(1 + c)^{3/2} > 1$. So $(1 + x)^{3/2} > 1 + \frac{3}{2}x$ for all $x > 0$.

5. Note that $(2,4)$ does lie on the given curve. We use implicit differentiation, the product rule and the chain rule to find:

$$2(x^2+y^2)(2x+2y\frac{dy}{dx}) = 50y + 50x\frac{dy}{dx}.$$

So in $(x,y) = (2,4)$ we have $40(4+8y'(2)) = 200+100y'(2)$, so $y'(2) = \frac{2}{11}$. The equation of the tangent line is therefore $y = 4 + \frac{2}{11}(x-2) = \frac{2}{11}x + \frac{40}{11}$.

6. a) Calculate the derivative: $f'(x) = 1 + e^x > 0$ for all x. So f is strictly increasing on $\mathbb{R}$ and therefore one-to-one on $\mathbb{R}$. Thus there exists an inverse-function $f^{(-1)}$. The domain of $f^{(-1)}$ is equal to the range of f. Since f is continuous and

$$\lim_{x\to -\infty} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to\infty} f(x) = \infty,$$

the range of f is $(-\infty,\infty)$ (Intermediate Value Theorem), which implies that the domain of $f^{(-1)}$ is also $(-\infty,\infty) = \mathbb{R}$.

b) Remark that $f(0) = 1$, so that $0 = f^{(-1)}(1)$. This yields

$$(f^{(-1)})'(1) = \frac{1}{f'(f^{(-1)}(1))} = \frac{1}{f'(0)} = \frac{1}{1+e^0} = \frac{1}{2}.$$

7. Introduce $f(x) = \arctan x$ and then calculate

$$\begin{cases} f(1) = \arctan 1 = \frac{\pi}{4}, & \\ f'(x) = \frac{1}{1+x^2}, & \text{so } f'(1) = \frac{1}{2}, \\ f''(x) = \frac{-2x}{(1+x^2)^2}, & \text{so } f''(1) = -\frac{1}{2}. \end{cases}$$

Therefore

$$P_2(x) = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2.$$

8. a) Use integration by parts:

$$\int \sqrt{x}\ln x\,dx = \frac{2}{3}x\sqrt{x}\ln x - \int \frac{2}{3}\sqrt{x}\,dx = \frac{2}{3}x\sqrt{x}\ln x - \frac{4}{9}x\sqrt{x} + C, C \in \mathbb{R}.$$

b) First use the substitution $t = e^x$ (so $dt = e^x dx$ or equivalently $dx = \frac{1}{t}dt$) to get:

$$\int_0^1 \frac{1}{e^x+1}\,dx = \int_1^e \frac{1}{t(t+1)}\,dt.$$

Now use the method of partial fractions:

$$\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1} = \frac{A(t+1)+Bt}{t(t+1)}.$$

Equate the coefficients of t and the constant term to obtain $A = 1$ and $B = -1$. So we find

$$\begin{aligned} \int_1^e \frac{1}{t(t+1)}\,dt &= \int_1^e \frac{1}{t} - \frac{1}{t+1}\,dt \\ &= \ln|t| - \ln|t+1|\Big|_1^e = 1 - \ln(e+1) + \ln 2. \end{aligned}$$

c) This is an improper integral, so:

$$\int_{-2}^{\infty} \frac{1}{x^2+4x+8}\,dx = \lim_{R\to\infty} \int_{-2}^{R} \frac{1}{(x+2)^2+4}\,dx =$$

$$\lim_{R\to\infty} \int_{-2}^{R} \frac{1/4}{((x+2)/2)^2+1}\,dx = \lim_{R\to\infty} \frac{1}{2} \arctan\left(\frac{x+2}{2}\right)\Big|_{-2}^{R} =$$

$$\lim_{R\to\infty} \frac{1}{2}\left(\arctan\left(\frac{R+2}{2}\right) - \arctan 0\right) = \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4}.$$

9. This is an improper integral of the first kind, but we cannot find an antiderivative easily. So we use a comparison test. Since

$$0 < \frac{\arctan x}{x\sqrt{x}} < \frac{\pi/2}{x\sqrt{x}}$$

and since

$$\int_1^{\infty} \frac{1}{x\sqrt{x}}\,dx \text{ is convergent } (p\text{-integral with } p = \tfrac{3}{2}),$$

the given integral is also convergent.