Second test Calculus 1, 24-10-2016, Solutions.

1. a) There are no boundary points and no singular points, so we only have to consider critical points, and the behavior of f when x tends to $-\infty$ or ∞. First calculate the derivative

$$f'(x) = 2xe^{-2x} - 2x^2e^{-2x} = 2x(1-x)e^{-2x}.$$

So $f'(x) = 0$ implies $x = 0$ or $x = 1$. Since $f'(x) < 0$ (so f is strictly decreasing) on $(-\infty, 0)$ and on $(1, \infty)$ and since $f'(x) > 0$ (so f is strictly increasing) on $(0, 1)$, f has an minimum in $x = 0$ with value $f(0) = 0$ and a maximum in $x = 1$ with value $f(1) = e^{-2}$. To investigate if these extreme values are local or absolute, calculate $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = \infty$ (both standard limits). So we find that 0 is the absolute minimum of f on $\mathbb{R}$ and that e^{-2} is a local maximum in $x = 1$.

b) Calculate

$$f''(x) = 2e^{-2x} - 4xe^{-2x} - 4xe^{-2x} + 4x^2e^{-2x} = 2(2x^2 - 4x + 1)e^{-2x},$$

so $f''(x) = 0$ implies that $x = 1 \pm \frac{1}{2}\sqrt{2}$. Since $f''(x) > 0$ on $(-\infty, 1 - \frac{1}{2}\sqrt{2})$ and on $(1 + \frac{1}{2}\sqrt{2}, \infty)$ and since $f''(x) < 0$ on $(1 - \frac{1}{2}\sqrt{2}, 1 + \frac{1}{2}\sqrt{2})$ the curve $y = f(x)$ has two inflection points with x-values $x = 1 + \frac{1}{2}\sqrt{2}$ and $x = 1 - \frac{1}{2}\sqrt{2}$.

2. a) Use the product-rule to find that

$$f'(x) = 1 + 2x \arctan(x) > 0 \text{ for all } x \in \mathbb{R},$$

(since $\arctan(x) < 0$ if $x < 0$ and $\arctan(x) > 0$ if $x > 0$, so $x \arctan(x) \geq 0$) so f is strictly increasing on $\mathbb{R}$ and therefore one-to-one on $\mathbb{R}$. So the inverse-function $f^{(-1)}$ exists. The domain of $f^{(-1)}$ is equal to the range of f. We know that f is continuous on $\mathbb{R}$ and that

$$\lim_{x\to-\infty} f(x) = \infty \times \left(-\frac{\pi}{2}\right) = -\infty \qquad \text{and} \qquad \lim_{x\to\infty} f(x) = \infty \times \frac{\pi}{2} = \infty,$$

so the range of f is $\mathbb{R}$, which implies that the domain of $f^{(-1)}$ is also $\mathbb{R}$.

b) Remark that $f(1) = \frac{\pi}{2}$, so that $f^{(-1)}(\pi/2) = 1$. This yields

$$(f^{(-1)})'(\pi/2) = \frac{1}{f'(f^{(-1)}(\pi/2))} = \frac{1}{f'(1)} = \frac{1}{1 + 2\arctan(1)} = \frac{1}{1 + \pi/2} = \frac{2}{2 + \pi}.$$

3. First take the natural logarithm of the expression. Then use l'Hospital's rule on the resulting expression, and divide numerator and denominator by e^{2x} in the next step:

$$\lim_{x\to\infty} \ln\left(e^{2x} + 3x\right)^{\frac{1}{4x}} = \lim_{x\to\infty} \frac{\ln\left(e^{2x} + 3x\right)}{4x}$$

$$\overset{(H)}{=} \lim_{x\to\infty} \frac{\frac{2e^{2x}+3}{e^{2x}+3x}}{4} = \lim_{x\to\infty} \frac{2 + 3e^{-2x}}{4(1 + 3xe^{-2x})} = \frac{2 + 0}{4(1 + 0)} = \frac{1}{2}.$$

So the original limit is $e^{\frac{1}{2}} = \sqrt{e}$.

4. a) Calculate

$$\begin{cases} f(1) = \sin(\pi) = 0, & \\ f'(x) = \pi\cos(\pi x), & \text{so } f'(1) = -\pi, \\ f''(x) = -\pi^2\sin(\pi x), & \text{so } f''(1) = 0, \\ f'''(x) = -\pi^3\cos(\pi x), & \text{so } f'''(1) = \pi^3. \end{cases}$$

Therefore

$$P_3(x) = 0 - \pi\frac{(x-1)}{1!} + 0\frac{(x-1)^2}{2!} + \pi^3\frac{(x-1)^3}{3!} = -\pi(x-1) + \frac{1}{6}\pi^3(x-1)^3.$$

b)

$$\begin{aligned} \lim_{x\to1}\frac{\sin(\pi x) + \pi x - \pi}{(x-1)^3} &= \lim_{x\to1}\frac{-\pi(x-1) + \frac{1}{6}\pi^3(x-1)^3 + O((x-1)^4) + \pi x - \pi}{(x-1)^3} \\ &= \lim_{x\to1}\frac{\frac{1}{6}\pi^3(x-1)^3 + O((x-1)^4)}{(x-1)^3} \\ &= \lim_{x\to1}\frac{1}{6}\pi^3 + O((x-1)^1) = \frac{1}{6}\pi^3. \end{aligned}$$

5. a) Use the substitution $t = \ln x$ (so $dt = \frac{1}{x}\,dx$):

$$\int \frac{\cos(\ln x)}{x}\,dx = \int \cos(t)\,dt = \sin(t) + C = \sin(\ln x) + C.$$

b) Factorize the denominator and use partial fraction expansion:

$$\begin{aligned} &\int_0^1 \frac{3x+2}{x^2-4}\,dx = \int_0^1 \frac{3x+2}{(x+2)(x-2)}\,dx \\ &= \int_0^1 \frac{1}{x+2} + \frac{2}{x-2}\,dx = \ln|x+2| + 2\ln|x-2|\Big|_0^1 \\ &= \ln 3 + 2\ln 1 - \ln 2 - 2\ln 2 = \ln 3 - 3\ln 2. \end{aligned}$$

6. a) Use integration by parts twice. Remember that $n \geq 2$, so that x^n and x^{n-1} are both zero for $x = 0$. Then we get

$$\begin{aligned} I_n &= -x^n\cos(x)\Big|_0^{\pi/2} + \int_0^{\pi/2} nx^{n-1}\cos(x)\,dx \\ &= nx^{n-1}\sin(x)\Big|_0^{\pi/2} - \int_0^{\pi/2} n(n-1)x^{n-2}\sin(x)\,dx \\ &= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}, \qquad \text{for } n \geq 2. \end{aligned}$$

b) First calculate $I_0 = \int_0^{\pi/2}\sin(x)\,dx = -\cos(x)\Big|_0^{\pi/2} = 1$. Then use the reduction formula for $n = 4$:

$$\begin{aligned} &I_4 = 4\left(\frac{\pi}{2}\right)^3 - 12I_2 = 4\left(\frac{\pi}{2}\right)^3 - 12\left(2\left(\frac{\pi}{2}\right)^1 - 2I_0\right) = \\ &= 4\left(\frac{\pi}{2}\right)^3 - 24\left(\frac{\pi}{2}\right) + 24 = \frac{1}{2}\pi^3 - 12\pi + 24. \end{aligned}$$

7. This is an improper integral of the first and second kind. So split into two parts (I_1 and I_2) and consider each part separately:

   (i) On $[0,1]$: $I_1 = \int\limits_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$. Since $\frac{e^{-x}}{\sqrt{x}} < \frac{1}{\sqrt{x}}$ and since $\int\limits_0^1 \frac{1}{\sqrt{x}}\,dx$ is convergent (p-integral with $p = \frac{1}{2}$), I_1 is also convergent.

   (ii) On $[1,\infty)$: $I_2 = \int\limits_1^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$. Since $\frac{e^{-x}}{\sqrt{x}} < e^{-x}$ and since

$$\int\limits_1^\infty e^{-x}\,dx = \lim_{R\to\infty}\int_1^R e^{-x}\,dx = \lim_{R\to\infty}\left(e^{-1} - e^{-R}\right) = e^{-1},$$

   so is convergent, I_2 is also convergent.

   Combining (i) and (ii) we conclude that the given improper integral is convergent.