Resit Calculus 1, 05-01-2016, Solutions.

1. a) The limit does not exist. Distinguish between $\lim_{x\to 0-}$ and $\lim_{x\to 0+}$ and use $\sqrt{x^2+x^3} = \sqrt{x^2}\sqrt{1+x} = |x|\sqrt{1+x}$ and the standard limit $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$ to find

$$\lim_{x\to 0-}\frac{\sqrt{x^2+x^3}}{\sin(x)} = \lim_{x\to 0-}\sqrt{1+x}\cdot\frac{-x}{\sin(x)} = -1,$$

while

$$\lim_{x\to 0+}\frac{\sqrt{x^2+x^3}}{\sin(x)} = \lim_{x\to 0+}\sqrt{1+x}\cdot\frac{x}{\sin(x)} = 1,$$

so the limits are not equal and so the original limit does not exist.

b) Multiply numerator and denominator by the expression $x - \sqrt{x^2+3x+1}$, then divide numerator and denominator by x and use $\sqrt{x^2} = -x$ for $x < 0$ to get

$$\lim_{x\to-\infty} x + \sqrt{x^2+3x+1} = \lim_{x\to-\infty} x + \sqrt{x^2+3x+1}\cdot\frac{x-\sqrt{x^2+3x+1}}{x-\sqrt{x^2+3x+1}}$$

$$= \lim_{x\to-\infty}\frac{-3x-1}{x-\sqrt{x^2+3x+1}} = \lim_{x\to-\infty}\frac{-3-\frac{1}{x}}{1+\sqrt{1+\frac{3}{x}+\frac{1}{x^2}}} = -\frac{3}{2}.$$

2. a) Since $f(0) = 1$ and $f'(x) = e^x\cos(x) - e^x\sin(x)$, so $f'(0) = 1$, we find $L(x) = f(0) + f'(0)(x-0) = 1 + x$.

b) There are no singular points, so we only have to consider critical points, and two boundary points. First $f'(x) = 0$ implies $\cos(x) = \sin(x)$, so $\tan(x) = 1$, with only solution $x = \frac{\pi}{4}$ on the given interval. Since $f'(x) > 0$ (so f is increasing) on $[-\frac{1}{2}\pi, \frac{\pi}{4}]$ and $f'(x) < 0$ (so f is decreasing) on $[\frac{1}{4}\pi, \frac{\pi}{2}]$, f has an absolute maximum in $x = \frac{\pi}{4}$ with value $\frac{1}{2}\sqrt{2}e^{\frac{\pi}{4}}$. And f has its absolute minimum value in the boundary points of the interval: $f(-\frac{\pi}{2}) = f(\frac{\pi}{2}) = 0$.

c) Calculate $f''(x) = -2e^x\sin(x)$, so $f''(x) = 0$ implies that $x = 0$ (on the given interval). Since $f''(x) > 0$ on $[-\frac{\pi}{2}, 0)$ and $f''(x) < 0$ on $(0, \frac{\pi}{2}]$ the curve $y = f(x)$ has an inflection point $(0, 1)$.

3. For continuity we must have:

$$\lim_{x\to 0} f_c(x) = f_c(0) = c.$$

We calculate the one-sided limits:

$$\lim_{x\to 0-} f(x) = c \quad\text{and}\quad \lim_{x\to 0+} f(x) = \sqrt{c}.$$

The equation $c = \sqrt{c}$ has solutions $c = 0$ and $c = 1$. So if we choose $c = 0$ or $c = 1$ the function is continuous at $x = 0$.

4. a) Calculate the derivative:

$$f'(x) = \frac{\sqrt{x^2+4} - \frac{2x^2}{2\sqrt{x^2+4}}}{x^2+4} = \frac{4}{(x^2+4)\sqrt{x^2+4}} > 0 \text{ for all } x.$$

So f is strictly increasing on $\mathbb{R}$ and therefore one-to-one on $\mathbb{R}$.

b) Since f is one-to-one, there exists an inverse-function $f^{(-1)}$. The domain of $f^{(-1)}$ is equal to the range of f. We know that f is continuous and increasing on $\mathbb{R}$ and that

$$\lim_{x \to -\infty} f(x) = -1 \quad \text{and} \quad \lim_{x \to \infty} f(x) = 1,$$

so the range of f is $(-1, 1)$, which implies that the domain of $f^{(-1)}$ is also $(-1, 1)$.

c) Remark that $f(0) = 0$, so that $0 = f^{(-1)}(0)$. This yields

$$(f^{(-1)})'(0) = \frac{1}{f'(f^{(-1)}(0))} = \frac{1}{f'(0)} = \frac{1}{\frac{4}{4\sqrt{4}}} = 2.$$

5. Note that $(1, 0)$ does lie on he given curve. We use implicit differentiation, the product rule and the chain rule to find:

$$\frac{dy}{dx} - \sin(y) \cdot \frac{dy}{dx} = 2x + \frac{1}{x}.$$

So the slope of the tangent line is:

$$\left.\frac{dy}{dx}\right|_{(1,0)} = \left.\frac{2x + \frac{1}{x}}{1 - \sin(y)}\right|_{(1,0)} = \frac{3}{1} = 3.$$

And therefore the equation of the tangent line is $y = 0 + 3(x - 1) = 3x - 3$.

6. If $f'(0)$ exists it must be equal to:

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h\sqrt{|h|}\sin(\ln|h|) - 0}{h} = \lim_{h \to 0} \sqrt{|h|}\sin(\ln|h|).$$

The last limit equals 0 and can be calculated using the squeeze theorem. Since

$$-\sqrt{|h|} \leq \sqrt{|h|}\sin(\ln|h|) \leq \sqrt{|h|}$$

and since $\pm\sqrt{|h|}$ tend to 0 if h tends to 0, we find $f'(0) = 0$.

7. We define $f(x) = \sin(x) + \cos(x) - 3x + 2$ which is a differentiable (thus continuous) function on $\mathbb{R}$. First we will prove that the equation $f(x) = 0$ has at least one solution. Consider the interval $[0, \pi]$. Then $f(0) = 2 > 0$ and $f(\pi) = 1 - 3\pi < 0$. Since f is continuous on $[0, \pi]$ the Intermediate Value Theorem implies that there exists a $c \in (0, \pi)$ where $f(c) = 0$.
Next we will prove that the equation $f(x) = 0$ has at most one solution. Therefore consider $f'(x) = \cos(x) - \sin(x) - 3 < 0$ for all x, so f is strictly decreasing on $\mathbb{R}$ and therefore we have at most one $d \in \mathbb{R}$ where $f(d) = 0$.
Combining both results proves that the equation has exactly one (real) solution.

8. a) Use the substitution $t = \sqrt{x}$ (so $dt = \frac{1}{2\sqrt{x}}\,dx$ and thus $dx = 2\sqrt{x}\,dt = 2t\,dt$) followed by integration by parts:

$$\int e^{\sqrt{x}}\,dx = \int 2te^t\,dt = 2te^t - \int 2e^t\,dt$$
$$= 2te^t - 2e^t + C = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.$$

b) This is an improper integral, so:

$$\int_1^\infty \frac{1}{x^3+x^2}\,dx = \lim_{R\to\infty}\int_1^R \frac{1}{x^3+x^2}\,dx = \lim_{R\to\infty}\int_1^R \frac{1}{x^2(x+1)}\,dx.$$

Now use the method of partial fractions:

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} = \frac{Ax(x+1)+B(x+1)+Cx^2}{x^2(x+1)}.$$

Equate the coefficients of x^2, x and the constant term to obtain respectively:

$$A + C = 0,\quad A + B = 0 \quad \text{and} \quad B = 1,$$

with solution $A = -1, B = 1$ and $C = 1$. So we find

$$\begin{aligned}
\int_1^\infty \frac{1}{x^3+x^2}\,dx &= \lim_{R\to\infty}\int_1^R -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\,dx \\
&= \lim_{R\to\infty} -\ln|x| - \frac{1}{x} + \ln|x+1|\Big|_1^R \\
&= \lim_{R\to\infty} -\ln(R) - \frac{1}{R} + \ln(R+1) + 1 - \ln 2 \\
&= \lim_{R\to\infty} \ln\left(\frac{R+1}{R}\right) - \frac{1}{R} + 1 - \ln 2 = 1 - \ln 2.
\end{aligned}$$