Second test Calculus 1, 19-10-2015, Solutions.

1. a) Use the product-rule to find that

$$f'(x) = \sqrt{x^2+3} + \frac{x^2}{\sqrt{x^2+3}} = \frac{2x^2+3}{\sqrt{x^2+3}} > 0 \text{ for all } x,$$

so f is strictly increasing on $\mathbb{R}$ and therefore one-to-one on $\mathbb{R}$.

b) Since f is one-to-one, there exists an inverse-function $f^{(-1)}$. The domain of $f^{(-1)}$ is equal to the range of f. We know that f is continuous on $\mathbb{R}$ and that

$$\lim_{x\to\,-\infty} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to\infty} f(x) = \infty,$$

so the range of f is $\mathbb{R}$, which implies that the domain of $f^{(-1)}$ is also $\mathbb{R}$.

c) Remark that $f(0) = 0$, so that $0 = f^{(-1)}(0)$. This yields

$$(f^{(-1)})'(0) = \frac{1}{f'(f^{(-1)}(0))} = \frac{1}{f'(0)} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}.$$

2. a) There are no singular points, so we only have to consider critical points, one boundary point and the behavior of f near $x = 0$. Calculate the derivative

$$f'(x) = 2x\ln(x) + x^2 \cdot \frac{1}{x} = x(2\ln(x) + 1).$$

So $f'(x) = 0$ implies $x = e^{-\frac{1}{2}}$. Since $f'(x) < 0$ (so f is decreasing) on $(0, e^{-\frac{1}{2}})$ and $f'(x) > 0$ (so f is increasing) on $(e^{-\frac{1}{2}}, e)$, f has an absolute minimum in $x = e^{-\frac{1}{2}}$ with value $-\frac{1}{2}e^{-1}$. For the absolute maximum we consider $f(e) = e^2$ and $\lim_{x\to 0+} f(x) = 0$ (standard limit). So we find that e^2 is the absolute maximum of f on $(0, e]$.

b) Calculate $f''(x) = 2\ln(x) + 3$, so $f''(x) = 0$ implies that $x = e^{-\frac{3}{2}}$. Since $f''(x) < 0$ on $(0, e^{-\frac{3}{2}})$ and $f''(x) > 0$ on $(e^{-\frac{3}{2}}, e)$ the curve $y = f(x)$ has an inflection point $(e^{-\frac{3}{2}}, -\frac{3}{2}e^{-3})$.

3. a) Use the fact that $\ln(x^2e^x) = \ln(x^2) + \ln(e^x) = 2\ln(x) + x$, so that

$$\lim_{x\to\infty}\Big(x^2 - \ln(x^2e^x)\Big) = \lim_{x\to\infty} x^2\Big(1 - \frac{2\ln(x)}{x^2} - \frac{1}{x}\Big) = \infty \cdot 1 = \infty,$$

since $\lim_{x\to\infty} \frac{2\ln(x)}{x^2} = 0$ (standard limit, or with l'Hospital).

b) Use $a^b = e^{b\ln(a)}$:

$$\lim_{x\to 0}\Big(1 + \sin(x)\Big)^{1/x} = \lim_{x\to 0} e^{\frac{\ln(1+\sin(x))}{x}} = e,$$

since l'Hospital ($\left[\frac{0}{0}\right]$-situation) gives

$$\lim_{x\to 0}\frac{\ln(1+\sin(x))}{x} = \lim_{x\to 0}\frac{\cos(x)}{1+\sin(x)} = 1.$$

4. a) Calculate $f(1) = 1$, $f'(x) = -\frac{1}{x^2}$, so $f'(1) = -1$, and $f''(x) = \frac{2}{x^3}$, so $f''(1) = 2$. Therefore $P_2(x) = 1 - (x-1) + (x-1)^2$.

b) First we need $f^{(3)}(x) = -\frac{6}{x^4}$. Taylor's theorem gives

$$f(x) = P_2(x) + E_2(x), \text{ with error-function } E_2(x) = \frac{f^{(3)}(s)}{3!}(x-1)^3,$$

with $1 < s < x$ or $x < s < 1$. We use $x = 1.02$, which gives the approximation $P_2(1.02) = 1 - 0.02 + (0.02)^2 = 0.9804$. The absolute value of the error is

$$|E_2(1.02)| = \left|\frac{-6/s^4}{6}(0.02)^3\right| = \left|\frac{(0.02)^3}{s^4}\right| < (0.02)^3,$$

since $1 < s < 1.02$, so $\frac{1}{1.02} < \frac{1}{s} < 1$ and $\frac{1}{(1.02)^4} < \frac{1}{s^4} < 1$.

5. With the Fundamental Theorem of Calculus, the chain-rule and the product-rule:

$$f'(x) = 3\int_4^{x^2} e^{-\sqrt{t}}\,dt + 3x \cdot e^{-\sqrt{x^2}} \cdot 2x,$$

So

$$f'(2) = 3\int_4^4 e^{-\sqrt{t}}\,dt + 6 \cdot e^{-2} \cdot 4 = 0 + 24e^{-2} = 24e^{-2}$$

6. a) Use the substitution $t = \sqrt{x}$ (so $dt = \frac{1}{2\sqrt{x}}\,dx$ and therefore $dx = 2\sqrt{x}\,dt = 2t\,dt$) followed by integration by parts:

$$\int \sqrt{x}\cos(\sqrt{x})\,dx = \int 2t^2\cos(t)\,dt = 2t^2\sin(t) - \int 4t\sin(t)\,dt$$
$$= 2t^2\sin(t) + 4t\cos(t) - \int 4\cos(t)\,dt = 2t^2\sin(t) + 4t\cos(t) - 4\sin(t) + C$$
$$= 2x\sin(\sqrt{x}) + 4\sqrt{x}\cos(\sqrt{x}) - 4\sin(\sqrt{x}) + C.$$

b) Factorize the denominator and use partial fraction expansion:

$$\int \frac{x}{x^2+x-20}\,dx = \int \frac{x}{(x+5)(x-4)}\,dx$$
$$= \int \frac{5/9}{x+5} + \frac{4/9}{x-4}\,dx = \frac{5}{9}\ln|x+5| + \frac{4}{9}\ln|x-4| + C.$$

c) This is an improper integral, so:

$$\int_5^\infty \frac{1}{x^2+25}\,dx = \lim_{R\to\infty}\int_5^R \frac{1}{x^2+25}\,dx = \lim_{R\to\infty}\int_5^R \frac{1/25}{(x/5)^2+1}\,dx =$$
$$\lim_{R\to\infty}\frac{1}{5}\arctan\left(\frac{x}{5}\right)\Big|_5^R = \lim_{R\to\infty}\frac{1}{5}\left(\arctan\left(\frac{R}{5}\right) - \arctan\left(\frac{5}{5}\right)\right) = \frac{1}{5}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{20}.$$