VU University Amsterdam, Faculty of Sciences.
Calculus 1, 6 January 2015, 18:30-21:15, solutions.

1. Remark that the inequality is equivalent to $\frac{4x-9}{2-x} \leq 0$. The fraction at the left-hand side in not defined if $x = 2$ and is equal to 0 if $x = \frac{9}{4}$. We make the following table:

x		2		$\frac{9}{4}$	
$2-x$	+	0	-	-	-
$4x-9$	-	-	-	0	+
$\frac{4x-9}{2-x}$	-	↯	+	0	-

So the solution is: $x \in (-\infty, 2) \cup [\frac{9}{4}, \infty)$.

2. a) $\lim_{x\to 0}(3x^2 - 3x + 1)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{\ln(3x^2-3x+1)}{x}}$. First calculate (with l'Hospital)

$$\lim_{x\to 0} \frac{\ln(3x^2 - 3x + 1)}{x} = \lim_{x\to 0} \frac{6x-3}{3x^2 - 3x + 1} = -3.$$

So $\lim_{x\to 0}(3x^2 - 3x + 1)^{\frac{1}{x}} = e^{-3}$.

b)

$$\lim_{x\to 0} \frac{\sqrt{9-x} - \sqrt{-3x+9}}{x} = \lim_{x\to 0} \left(\frac{\sqrt{9-x} - \sqrt{-3x+9}}{x} \cdot \frac{\sqrt{9-x} + \sqrt{-3x+9}}{\sqrt{9-x} + \sqrt{-3x+9}} \right)$$
$$= \lim_{x\to 0} \frac{2x}{x(\sqrt{9-x} + \sqrt{-3x+9})} = \lim_{x\to 0} \frac{2}{\sqrt{9-x} + \sqrt{-3x+9}} = \frac{2}{\sqrt{9} + \sqrt{9}} = \frac{1}{3}.$$

3. a) With implicit differentiation we find $y' = \cos(3xy) \cdot (3y + 3xy')$.
So $y' = \dfrac{3y\cos(3xy)}{1 - 3x\cos(3xy)}$.

b)

$$y'(\frac{\pi}{9}) = \frac{3 \cdot \frac{1}{2} \cdot \cos(3 \cdot \frac{1}{9}\pi \cdot \frac{1}{2})}{1 - 3 \cdot \frac{1}{9}\pi \cdot \cos(3 \cdot \frac{1}{9}\pi \cdot \frac{1}{2})} = \frac{\frac{3}{4}\sqrt{3}}{1 - \frac{1}{3}\pi \cdot \frac{1}{2}\sqrt{3}} = \frac{9\sqrt{3}}{12 - 2\pi\sqrt{3}}.$$

So the equation of the tangent line is given by $y = \dfrac{9\sqrt{3}}{12 - 2\pi\sqrt{3}}(x - \dfrac{\pi}{9}) + \dfrac{1}{2}$.

4. a) For continuity we need $\lim_{x\to 0} f(x) = f(0) = b$. We have $\lim_{x\to 0^-} f(x) = b$ and (using l'Hospital) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\ln(1+3x)}{2x} \overset{l'H}{=} \lim_{x\to 0^+} \frac{3}{2(1+3x)} = \frac{3}{2}$. So $b = \frac{3}{2}$ and a can be any real number.

b) If f is differentiable, then f also continuous. So we already know that $b = \frac{3}{2}$. According to the definition f is differentiable in 0 if and only if $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ exists and in that case $f'(0)$ is equal to that limit. Remark that:

$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{\ln(1+3h) - 3h}{2h^2} \overset{l'H}{=} \lim_{h\to 0^+} \frac{\frac{3}{1+3h} - 3}{4h} = -\frac{9}{4}.$$

Furthermore: $\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = a$. So $a = -\frac{9}{4}$.

5. There is more than one solution method, but here we use the definition

$$\begin{cases} f(-2) = e^{-4} & \\ f'(x) = 2e^{2x}, & \text{so } f'(-2) = 2e^{-4} \\ f''(x) = 4e^{2x}, & \text{so } f''(-2) = 4e^{-4} \\ f'''(x) = 8e^{2x}, & \text{so } f'''(-2) = 8e^{-4} \end{cases}$$

This gives

$$P_3(x) = e^{-4} + 2e^{-4}(x+2) + 2e^{-4}(x+2)^2 + \frac{4}{3}e^{-4}(x+2)^3.$$

6. Take an arbitrary $x > 0$. Consider $f(t) = \sqrt{2+3t}$ on the interval $[0, x]$. The function is continuous there and differentiable on the open interval $(0, x)$. Remark that $f'(t) = \frac{3}{2\sqrt{2+3t}}$. The Mean Value Theorem says that there exists a $c \in (0, x)$ such that:

$$\frac{f(x) - f(0)}{x - 0} = \frac{\sqrt{2+3x} - \sqrt{2}}{x} = f'(c) = \frac{3}{2\sqrt{2+3c}} < \frac{3}{2\sqrt{2}}.$$

So $\sqrt{2+3x} - \sqrt{2} < \frac{3x}{2\sqrt{2}}$. So $\sqrt{2+3x} < \sqrt{2} + \frac{3x}{2\sqrt{2}}$.

7. With the fundamental theorem of Calculus we find that:

$$f'(x) = 2\int_{\frac{\pi}{4}}^{\arctan(x)} \tan^2(t)\,dt + 2x \cdot \frac{1}{1+x^2}\tan^2(\arctan x).$$

So $f'(1) = 2 \cdot 0 + \frac{2 \cdot 1 \cdot 1^2}{1+1^2} = 1$.

8. a) Substitute $u = 3x^2 + 1$. Then $du = 6x\,dx$.

$$\int_0^1 \frac{x}{\sqrt{3x^2+1}}\,dx = \frac{1}{6}\int_1^4 \frac{1}{\sqrt{u}}du = \frac{1}{6}[2\sqrt{u}]_1^4 = \frac{1}{3}.$$

b) Use integration by parts twice. Take $U = x^2$ and $dV = e^x\,dx$. Then $dU = 2x\,dx$ and $V = e^x$. So

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx.$$

Now take $U = x$ and $dV = e^x\,dx$. Then $dU = dx$ and $V = e^x$. So

$$x^2 e^x - 2\int x e^x\,dx = x^2 e^x - 2x e^x + 2\int e^x\,dx = x^2 e^x - 2x e^x + 2e^x + C$$

c) Remark that $\displaystyle\int \frac{x^2}{x^2+5x+6}\,dx = \int \left(1 - \frac{5x+6}{x^2+5x+6}\right)dx.$

Furthermore $x^2 + 5x + 6 = (x+2)(x+3)$. So: $\frac{5x+6}{x^2+5x+6} = \frac{A}{x+2} + \frac{B}{x+3}$ for some A and B. This gives $A + B = 5$ and $3A + 2B = -9$. So $A = -4$ en $B = 9$. Finally:

$$\int \frac{x^2}{x^2+5x+6}\,dx = \int \left(1 + \frac{4}{x+2} - \frac{9}{x+3}\right)dx = x + 4\ln|x+2| - 9\ln|x+3| + C.$$

9. The integral is divergent, because $\frac{1+\ln x}{x} > \dfrac{1}{x}$ and $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ is divergent.