SAMPLE EXAM

Problem 1 (20 points)**.**

State for each of the claims below whether it is *true* or *false*. **NOTE:** You do **not** need to justify or prove your answers here.

(a) Let $I = (G = (V,A), (\ell_a)_{a\in A}, (s_i,t_i)_{i\in[k]}, (r_i)_{i\in[k]})$ be an instance of the selfish routing game with standard latency functions. A feasible flow f for I is a Wardrop flow if
$$\forall i \in [k],\ \forall P,Q \in \mathcal{P}_i,\ f_P > 0: \quad \ell_P(f) \geq \ell_Q(f).$$

(b) Let f be a Nash flow for a selfish routing instance I and define for every commodity $i \in [k]$, $c_i(f) = \min_{P\in\mathcal{P}_i} \ell_P(f)$. Then $c_i(f) = c_j(f)$ for all $i, j \in [k]$.

(c) There is an instance I of the selfish routing game with linear latency functions, i.e., for all $a \in A$, $\ell_a(x) = q_a x$ with $q_a > 0$, whose price of anarchy is $\frac{4}{3}$.

(d) Given an instance of the connection game, the social cost of every pure Nash equilibrium is at least H_n times the optimal social cost, where n is the number of players.

(e) A finite strategic game Γ has the finite improvement property if the transition graph $G(\Gamma)$ contains no directed cycles.

(f) The problem of computing a pure Nash equilibrium for symmetric network congestion games is in P.

(g) Let $\Pi_1 \in$ PLS and let Π_2 be PLS-complete. If Π_2 is PLS-reducible to Π_1 then Π_1 is PLS-complete.

(h) The price of anarchy of second-price auctions is bounded.

(i) Given an arbitrary matching market $(B, S, (v_{ik}))$, there always exist market-clearing prices.

(j) For generalized second-price auctions, bidding truthfully is a dominant strategy for every player.

Problem 1 (15 points). Let $I = (G = (V,A), (\ell_a)_{a\in A}, (s_i,t_i)_{i\in[k]}, (r_i)_{i\in[k]})$ be a selfish routing instance.

(a) Fix some integer $d \geq 0$ and assume that all latency functions are *monomials of degree d*, i.e., for every arc $a \in A$, $\ell_a(x) = q_a x^d$ for some $q_a \geq 0$. Derive a tight bound on the price of anarchy for these games.

(b) Suppose all latency functions are affine, i.e., for every arc $a \in A$, $\ell_a(x) = p_a x + q_a$ for some $p_a, q_a \geq 0$. We say that a flow f is *α-fair* with $\alpha \geq 1$ if for every commodity i the latency of every flow-carrying s_i,t_i-path is at most α times larger than the minimum latency, i.e.,

$$\forall i \in [k],\ \forall P \in \mathcal{P}_i,\ f_P > 0: \quad \ell_P(f) \leq \alpha \cdot \min_{Q\in\mathcal{P}_i} \ell_Q(f)$$

Prove that every optimal flow is 2-fair and provide an example that shows that this is tight.

Problem 2 $(5+5+10+10$ points$)$.

Consider the following *scheduling game*: We are given a set of *jobs* $N = [n]$ that need to be processed on a set of *machines* $M = [m]$. Every job $j \in N$ has a *processing time* $p_j > 0$, which defines the amount of time that j needs to be processed. A *schedule* $\sigma = (\sigma_1, \dots, \sigma_n) \in M^n$ assigns each job $j \in N$ to a machine $\sigma_j \in M$ on which it is processed. The *load* $L_i(\sigma)$ of a machine $i \in M$ with respect to a given schedule σ is defined as the total processing time of all jobs that are assigned to i, i.e.,

$$L_i(\sigma) = \sum_{j\in N:\sigma_j=i} p_j.$$

Define the *completion time* $c_j(\sigma)$ of a job $j \in N$ with respect to a given schedule σ as the load of the machine to which job j is assigned, i.e., $c_j(\sigma) = L_i(\sigma)$ with $i = \sigma_j$. Suppose each job $j \in N$ corresponds to a selfish player who chooses a machine $\sigma_j \in M$ such that her own completion time is minimized. Define the *social cost* $C_{\max}(\sigma)$ of a schedule σ as the maximum load of a machine, i.e., $C_{\max}(\sigma) = \max_{i\in M} L_i(\sigma)$. A schedule σ^* that minimizes $C_{\max}$ is said to be *optimal*.

(a) Consider a scheduling game with $m = 2$ machines and $n = 4$ jobs. Let $p_1 = p_2 = 2$ and $p_3 = p_4 = 1$. Determine the price of anarchy for this instance.

(b) Generalize the example in (a) to show that for every $m \geq 2$ the price of anarchy of scheduling games is at least $2m/(m+1)$.

(c) Show that the price of anarchy for scheduling games is at most 2.

(d) Prove that pure Nash equilibria always exist in scheduling games. (Hint: Define $\Phi(\sigma) = (L_1(\sigma), \dots, L_m(\sigma)) \in \mathbb{R}^m$ as the *ordered* vector of machine loads such that $L_1(\sigma) \geq L_2(\sigma) \geq \cdots \geq L_m(\sigma)$. Show that Φ is a generalized ordinal potential function with respect to the lexicographic ordering.)

Problem 3 $(5+10$ points$)$**.**

Consider a single-item auction with player set $N=[n]$. Each player $i \in N$ has a private valuation v_i and specifies a bid b_i.

(a) In a *first-price auction* the item is given to a player whose bid is largest (ties are broken arbitrarily) at a price equal to the bid of this player. Show that the first-price auction is not strategyproof.

(b) Show that in a Vickrey auction a player i might be strictly worse of by bidding $b_i \neq v_i$ than by bidding truthfully. That is, show that for every player $i \in N$ and for every bid $b_i \neq v_i$ there is a bidding profile b_{-i} of the other players such that $u_i(b_{-i}, b_i) < u_i(b_{-i}, v_i)$.

Problem 4 $(10+5+5$ points$)$**.**

Consider the generalized second-price auction setting with n players and $m=n$ slots. Recall that the bids $b=(b_i)_{i\in N}$ constitute a pure Nash equilibrium if no player can increase her utility by unilaterally changing her bid.

(a) Show that the pure Nash equilibrium conditions can be expressed by $n-1$ inequalities for each player that must be satisfied.

We say that the bids $b=(b_i)_{i\in N}$ are *envy-free* if for every player $i \in N$ assigned to slot k (i.e., $i=\pi(k)$) and every other slot $j \neq k$

$$\alpha_k(v_i - b_{\pi(k+1)}) \geq \alpha_j(v_i - b_{\pi(j+1)}).$$

(The interpretation of "envy-free" here is that if we consider the prices for the slots to be fixed, then every player i is as happy getting her current slot at the current price as she would be getting any other slot at that slot's price.)

(b) Prove that if the bids $b=(b_i)_{i\in N}$ are envy-free then they constitute a pure Nash equilibrium.

(c) Give an example showing that there are bids $b=(b_i)_{i\in N}$ which constitute a pure Nash equilibrium but are not envy-free.