Exam Asymptotic Statistics

December 18, 2009

15.15–18.00

National Master's program Mastermath
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1. a) Consider a sequence of random variables (X_n) and some constant $c \in \mathbb{R}$. Show (directly from the definition of the appropriate modes of convergence) that $X_n \rightsquigarrow c \Rightarrow X_n \xrightarrow{\mathrm{P}} c$.

   b) Consider a sequence of random variables (X_n). Give a precise definition of uniform tightness. Show that $\mathrm{E}X_n^2 = O(1)$ implies uniform tightness of the sequence (X_n).

   c) Let X_n be the minimum of a random sample $Y_1, \dots, Y_n$ from the density $f(x) = x^2 + 2/3$ on $[0,1]$. Find a constant a and a sequence b_n such that $b_n(X_n - a)$ converges in distribution to a nondegenerate limit. Determine the limit law.

   d) We define the bias of the estimator $\hat{\theta}$ for estimating θ to be $\mathrm{E}(\hat{\theta} - \theta)$. Suppose that two different estimation procedures give two biased sequences of estimators T_n and S_n, with bias e_n and $-3e_n$, respectively, for some sequence $e_n \to 0$. The sequences of estimators are known to be asymptotically consistent. Find a linear combination of the sequences of estimators that give an unbiased sequence of estimators. Is it consistent?

2. Let Z be a three dimensional standard normal vector. Consider $X = (1,1,1)^T + LZ$, where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ a & \sqrt{2-a^2} & 0 \\ 0 & 0 & \sqrt{a} \end{bmatrix}.$$

   Does there exist an a such that the random variables $X_1 + X_2 + X_3$ and X_2 are independent? If yes, give this a.

3. Let $X_1, \dots, X_n$ be a sample from the normal distribution with mean θ and variance θ.

   a) Find a variance stabilizing transformation for the sample mean and construct a confidence interval for θ based on this.

   b) Find the limit distribution of the sequence $\sqrt{n}(\cos(\bar{X}_n) - \cos(\theta))$. For which values of θ is this distribution degenerate?

4. Let $X_1, X_2, \dots$ be independent and identically distributed random variables with probability density function

$$p_\mu(x) = \mu e^{-\mu x} \quad \text{for } x > 0.$$

   The distribution is exponential with unknown, but fixed, scale $\mu > 0$. Define, for $\theta > 0$, the function

$$\psi_\theta(x) = \frac{1}{\theta} - x \quad \text{for } x > 0,$$

and consider the Z-estimator $\hat{\theta}_n$ defined as a zero of the function

$$\Psi_n(\theta) = \mathbb{P}_n\psi_\theta = \frac{1}{n}\sum_{i=1}^{n}\psi_\theta(X_i).$$

a) Using properties of the function Ψ_n, show that $\hat{\theta}_n$ is well defined, *i.e.* Ψ_n has exactly one point where it takes the value zero.

b) Show that $\Psi_n(\theta)$ converges in probability, for fixed $\theta > 0$.

c) Prove that $\hat{\theta}_n \xrightarrow{\text{P}} \mu$.

d) The random variables $\sqrt{n}(\hat{\theta}_n - \mu)$ are asymptotically normally distributed. What parameters do you expect for this asymptotic normal distribution?

e) Derive the maximum likelihood estimator for μ. How does it relate to $\hat{\theta}_n$? Based on this and d), give the Fisher information I_μ.

5. Suppose that $X_1, \ldots, X_n$ is a random sample from the uniform distribution on $[0, \theta]$ for $\theta > 0$. Fix $t \in (0, \theta)$. Consider two estimators of $\text{P}(X_1 \leq t)$: $\mathbb{F}_n(t)$ and $T_n(t) = t/(2\bar{X}_n)$.

a) What is the asymptotic variance of the estimator $\mathbb{F}_n(t)$ of $\text{P}(X_1 \leq t)$? In other words, find the variance of the asymptotic distribution of $\sqrt{n}(\mathbb{F}_n(t) - \text{P}(X_1 \leq t))$.

b) What is the asymptotic variance of the estimator $T_n(t) = t/(2\bar{X}_n)$ of $\text{P}(X_1 \leq t)$? In other words, find the variance of the asymptotic distribution of $\sqrt{n}(T_n(t) - \text{P}(X_1 \leq t))$.

c) Compare the variances of $\mathbb{F}_n(t)$ and $T_n(t)$. For which values of t does the first estimator have a smaller variance?

1a	1b	1c	1d	2	3a	3b	4a	4b	4c	4d	4e	5a	5b	5c
2	2	2	2	5	3	3	2	1	3	3	2	2	2	2

The final grade is (number of points + 4)/4
Good luck with the exam!