Exam Asymptotic Statistics

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18.30–21.15

National Master's program Mastermath
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1. a) Let X_n be the maximum of a random sample $Y_1, \ldots, Y_n$ from the density $3(1-x)^2$ on $[0,1]$. Find constants a and b_n such that $b_n(X_n - a)$ converges in distribution to a nondegenerate limit. What is the cumulative distribution function of this limit distribution?

   b) Consider two sequences of random variables (X_n) and (Y_n) such that $X_n \rightsquigarrow N_1$ and $Y_n \rightsquigarrow N_2$ for two independent standard normal random variables N_1 and N_2. Prove or disprove (by counterexample) that this implies

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \rightsquigarrow \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}.$$

   c) Find an example of a sequence of random variables such that $X_n \to^P 0$ but $EX_n \equiv 0$. Show that your example indeed has these properties.

2. Consider three independent sequences of i.i.d. random variables $X_1, X_2, \ldots; Y_1, Y_2, \ldots$ and $Z_1, Z_2, \ldots$. The variance of all these variables is one. The hypothesis is that the expectations of all variables are one. Find the asymptotic distribution of the following (test) statistic, under the null hypothesis:

$$T_n = n(\bar{X}_n - 1)^2 + n(\bar{Y}_n - 1)^2 + n(\bar{Z}_n - 1)^2.$$

3. Let $X_1, \ldots, X_n$ be a random sample from the exponential distribution with mean $\theta > 0$. This means X_i has density

$$f_\theta(x) = \frac{1}{\theta} e^{-x/\theta} 1_{[0,\infty)}(x).$$

   a) Find a variance stabilizing transformation for the sample mean and construct an asymptotic 95% confidence interval for θ based on this.

   b) Consider the following two estimators for $P(X_1 > 1) = e^{-1/\theta}$:

$$T_n = \frac{1}{n}\sum_{i=1}^n 1_{(1,\infty)}(X_i), \text{ and } S_n = e^{-1/\bar{X}_n}.$$

   Derive the asymptotic distribution of these estimators.

4. Consider the location family of normal distributions with unit variance:

$$p_\theta(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\theta)^2/2} \quad x \in I\!R.$$

Let $X_1, X_2, \ldots$ be independent and normally distributed random variables with density p_{θ_0}, where θ_0 denotes the 'true parameter'. Define, for $\theta \in \mathbb{R}$, the function

$$\psi_\theta(x) = (x-\theta)^3 \quad (x \in \mathbb{R})$$

and consider the Z-estimator $\hat{\theta}_n$ defined as a zero of the function

$$\Psi_n(\theta) = \mathbb{P}_n \psi_\theta = \frac{1}{n}\sum_{i=1}^{n} \psi_\theta(X_i)$$

a) Using properties of the function Ψ_n, show that $\hat{\theta}_n$ is well defined. In other words: show that Ψ_n has exactly one point where it becomes zero.

b) Prove that $\hat{\theta}_n \to^P \mu$.

You may use that $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^6 e^{-x^2/2}\, dx = 15$.

c) The random variables $\sqrt{n}(\hat{\theta}_n - \theta_0)$ are asymptotically normally distributed. Use the parameter you expect for the asymptotic variance to compute the asymptotic relative efficiency of $\hat{\theta}_n$ and the sample mean $\bar{X}_n$

5. a) Given a sample $X_1, \ldots, X_n$ from a distribution with probability density f, give the definition of the kernel estimator $\hat{f}_{n,h}$ based on this sample and show that the Mean Squared Error (MSE) of $\hat{f}_{n,h}(x)$ decomposes in a bias- and variance term.

b) Formulate the Glivenko Cantelli theorem and give an outline of its proof (line of thought, basic ingredients).

1a	1b	1c	2	3a	3b	4a	4b	4c	5a	5b
4	3	3	3	4	3	2	2	3	3	2

The final grade is (number of points + 4)/3.6

Good luck with the exam!