Exam Asymptotic Statistics

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1. a) Let X_n be the maximum of a random sample $Y_1, \ldots, Y_n$ from the density $2(1-x)$ on $[0,1]$. Find constants a and b_n such that $b_n(X_n - a)$ converges in distribution to a nondegenerate limit. What is the cumulative distribution function of this limit distribution?

   b) Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two independent random samples from distributions with means and variances equal to (μ_1, σ_1^2) and (μ_2, σ_2^2) respectively. For $S_{X,n}^2, S_{Y,n}^2$ the sample variances of the two samples, let

$$S_n^2 = \frac{S_{X,n}^2 + S_{Y,n}^2}{n} \qquad \text{and} \qquad T_n = \frac{\bar{X}_n - \bar{Y}_n}{S_n}.$$

   Let $\xi_{\alpha/2}$ the upper $\alpha/2$-quantile of the standard normal distribution. Show that the test that rejects $H_0 : \mu_1 = \mu_2$ if $|T_n| > \xi_{\alpha/2}$ is of asymptotic level α. Hint: First show that $T_n \rightsquigarrow N(0,1)$ if the null hypothesis holds.

   c) Find an example of a sequence of random variables such that $X_n \rightsquigarrow 0$, but $EX_n \to \infty$. Show that your example indeed satisfies these properties.

2. a) Suppose that X_m and Y_n are independent with binomial(m,p) and binomial(n,p) distributions. Find the limit distribution of

$$C_{m,n}^2 = \frac{(X_m - mp)^2}{mp(1-p)} + \frac{(Y_n - np)^2}{np(1-p)}$$

   as $m, n \to \infty$.

   b) Let $X_1, \ldots, X_n$ be i.i.d. with finite fourth moment. Find constants a, b and c_n such that the sequence $c_n(\bar{X}_n - a, \overline{X_n^2} - b)$ converges in distribution to a nondegenerate limit. Determine the limit distribution. Here $\bar{X}_n$ and $\overline{X_n^2}$ are the averages of the X_i and the X_i^2 respectively.

3. a) Let $X_1, \ldots, X_n$ be a random sample from the Poisson distribution with mean θ. Find a variance stabilizing transformation for the sample mean and construct an asymptotic 95% confidence interval for θ based on this.

   b) Find the joint limit distribution of $\left(\sqrt{n}(\bar{X}_n - \mu), \sqrt{n}(S_n^2 - \sigma^2)\right)$ if $\bar{X}_n$ and S_n^2 are based on a sample of size n from a distribution with finite fourth moment. Under what condition on the underlying distribution are $\sqrt{n}(\bar{X}_n - \mu)$ and $\sqrt{n}(S_n^2 - \sigma^2)$ asymptotically independent?

4. Let $X_1, \ldots, X_n$ be independent $N(\mu, 1)$-distributed random variables. Define $\hat{\theta}_n$ as the point of minimum of the function $\theta \to \sum_{i=1}^n (X_i - \theta)^4$.

   a) Show that $\hat{\theta}_n \xrightarrow{P} \theta_0$ for some θ_0. Which θ_0 ?

   b) Show the asymptotic normality of the M-estimator $\hat{\theta}_n$.

   c) Determine the asymptotic relative efficiency of $\hat{\theta}_n$ with respect to the sample mean. Based on this result, what can you conclude about the quality of the estimator $\hat{\theta}_n$?
   Hint: the even moments of a standard normal random variable Z are given by $EZ^{2k} = (2k)!/(2^k k!)$.

Please turn over

5. a) It can be shown that the Mean Integrated Squared Error (MISE) of the kernel estimator $\hat{f}_{n,h}$ for the density f has the following form:

$$\mathrm{MISE}_f(\hat{f}_{n,h}) = I_{f,K}(n,h) + J_{f,K}(h).$$

The two terms (variance and squared bias) have the following asymptotic behavior:

$$nhI_{f,K}(n,h) \to c_1(f,K) > 0 \text{ and } h^{-4}J_{f,K}(h) \to c_2(f,K) > 0$$

for $n \to \infty$, $h \downarrow 0$ and $nh \to \infty$. We decide to choose $h = h_n = cn^{-\alpha}$ for some $0 < \alpha < 1$. Determine (and argue) which choice of α is 'asymptotically MISE optimal'.

b) Taking the optimal α, one can show that with $h = h_n = cn^{-\alpha}$,

$$\lim_{n\to\infty} n^{4\alpha}\mathrm{MISE}_f(\hat{f}_{n,h_n}) = \frac{c_1(f,K)}{c} + c^4 \cdot c_2(f,K).$$

Derive the optimal choice of c.

1a	1b	1c	2a	2b	3a	3b	4a	4b	4c	5a	5b
3	4	3	3	4	3	4	2	2	3	3	2

The final grade is (number of points + 4)/4.
Good luck with the exam!