Investments 4.1
Course code: 60412040

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Sample Exam!

Parts:	This sample exam contains 20 parts. In a real exam, each part would yield roughly the same number of points.
Grading:	(Does not apply)
Results:	(Does not apply)
Inspection:	(Does not apply)
Remark:	**Be complete, but concise!** Provide complete answers, including computations where appropriate. On verbal questions: always provide motivation/explanation of your answer in terms of the economic mechanism at play. A short "yes" or "no" will never do as an answer. But be concise in your answers, otherwise you'll loose to much time writing it down.

Notice formula sheet at the end!

1. An investor who is a mean-variance optimizer wants to allocate her portfolio optimally between two stocks and risk free bonds. The returns to the two stocks are jointly normally distributed with $R_{t+1} \sim \mathcal{N}(\mu, \Omega)$. Expected returns are given by

$$R^f = 1.05 \qquad E_t[R_{t+1}] = \begin{bmatrix} 1.15 \\ 1.10 \end{bmatrix}$$

The return of each risky asset has a standard deviation of 40 % and their returns are uncorrelated. The investor chooses a vector of portfolio weights w for the two stocks to maximize:

$$w' E_t[R_{t+1} + (1 - w'\iota)R^f] - \frac{\lambda}{2} w'\Omega w$$

   (a) Show (using matrix algebra) the optimal portfolio weights w for the risky assets are given by

$$\frac{1}{\lambda}\Omega^{-1}(E_t[R_{t+1}] - R^f\iota)$$

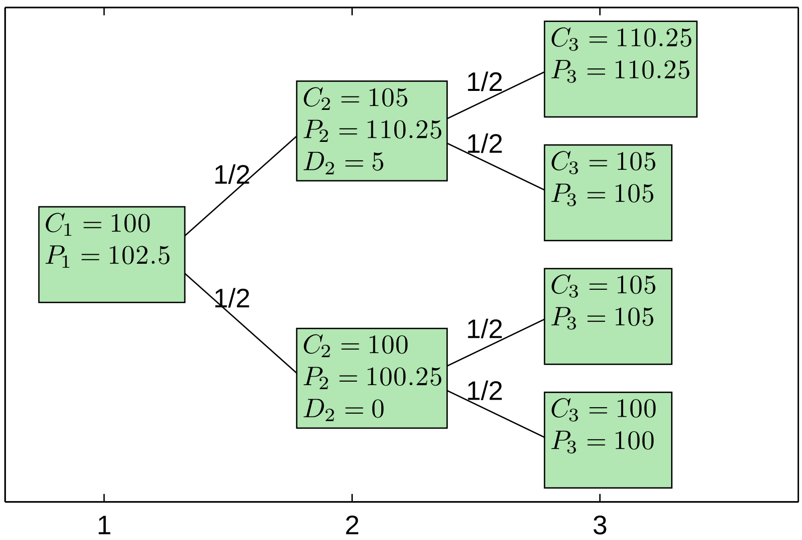
   (b) Compute the optimal portfolio weights assuming the investor has $\lambda = 2$.

   (c) Compute the expected return and standard deviation of the investors portfolio.

   (d) Say in words what would happen to the expected return and standard deviation of the investors portfolio if, starting from the optimal weights, the investor reduced the allocation to second risky asset by 1% and invested this money in the first risky asset.

   (e) Assume that every investor optimizes in the same way as the investor above, and that the two risky securities are the only ones traded. State why we know the CAPM holds. (Hint: which conditions are met?)

2. Consider the following simple economy. The probability of going up and down is equal to 50% always. P_t is the price of a stock in year t and C_t is consumption in year t, both depending on the state of the economy.



   (a) Assume the representative investor has a period utility function given by

$$U(C) = 2\sqrt{C} \qquad [= 2C^{0.5}]$$

   And a time discount factor of $\theta = 0.99$. Find the discount factor m_{t+1} at each node in years $t = 2$ and $t = 3$.

   (b) Use your result to find the price of a one and two year discount bond in year 1.

   (c) Find the price in year t of an Arrow-Debreu security that pays out 1 unit of consumption at the lowest node in year $t = 3$.

(d) $V(t, P_t)$ is the value of an American call option with strike price $K = 100$ which matures at $t = 3$. Find the value of the option at each node of the tree. (Remember to check for early exercise.)

3. The log return to a particular stock is given by

$$r_t = 0.05 + \epsilon_t - \frac{1}{2}\epsilon_{t-1} \qquad \epsilon_t = \mathcal{N}(0, \sigma_1^2) \tag{1}$$

and ϵ_t I.I.D.

(a) Give an example of irrational investor behavior that would give rise to such a return pattern.

(b) Compute

$$\frac{E[r_{t+1}]}{\text{var}\,(r_{t+1})} \qquad \text{and} \qquad \frac{E[r_{t,t+2}]}{\text{var}\,(r_{t,t+2})}$$

As well as the VR_1 and VR_2 statistics.

(c) How would you expect the average static allocation to equity differ between a 1 year and a 2 year investors based on your result above. Provide an economic explanation. (Assume both investors have a power utility function.)

The Campbell-Shiller decomposition of unexpected stock returns into dividend and discount rate news:

$$v_{t+1} = \eta_{d,t+1} - \eta_{r,t+1}$$

Is given by

$$r_{t+1} - E_t[r_{t+1}] = (E_{t+1} - E_t)\sum_{j=0}^{\infty} \rho^j \Delta d_{t+1+j} - (E_{t+1} - E_t)\sum_{j=1}^{\infty} \rho^j r_{t+1+j}$$

(d) Give a simplified expression for $\eta_{r,t+1}$ if returns are given by the process above.

Assume log dividend growth is given by

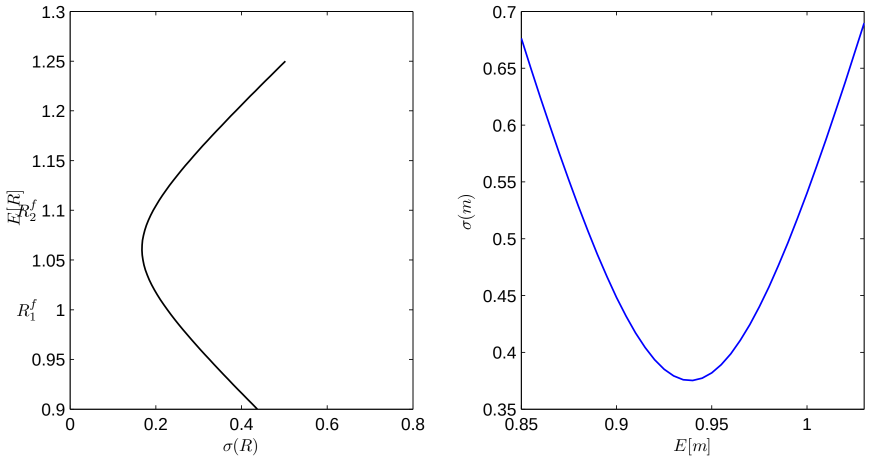
$$\Delta d_{t+1} = 0.02 + 0.2\Delta d_t + \nu_{t+1} \qquad \nu_{t+1} = \mathcal{N}(0, \sigma_2^2)$$

and ν_t I.I.D.

(e) Give a simplified expression for $\eta_{d,t+1}$ if returns are given by the process above.

(f) (Difficult, maybe) Express ϵ_t in terms of ϵ_{t-1} and ν_t

4. The left panel in this figure gives the Risky-Asset Frontier constructed using the expected returns and the covariance matrix of some securities returns. You are not sure what the risk free rate R^f is. The right hand panel gives the corresponding Hansen-Jagganathan bound. (HJ-Bound)



(a) Explain either in words or formulas how the Risky-Asset Frontier is constructed.

(b) In the left panel, sketch the mean-variance frontiers for $R^f = 1$ and $R^f = 1.1$. (Use different colors or line patterns for the two frontiers.)

(c) In the right panel, mark the two points on the Hansen-Jagannathan bound that corresponds to $R^f = 1$ and $R^f = 1.1$. Explain the link in either words or mathematical formulas.

(d) Explain in words or make a sketch of what would happen to the mean-variance frontier and the HJ-bound if you got access to data on the expected return and its covariance with the other returns of one more risky security.

(e) How would the HJ-bound simplify if you knew the risky free rate?

Vector derivatives

$$\frac{\partial x'a}{\partial a} = \frac{\partial a'x}{\partial a} = x$$
$$\frac{\partial x'Ax}{\partial x} = (A + A')x$$

Inverses

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}^{-1} = \begin{bmatrix} 1/a & 0 \\ 0 & 1/d \end{bmatrix} \qquad \text{(Special case)}$$

Expectations, variances

$$E_t[a + bx] = a + bE[x] \qquad \text{(Linearity)}$$
$$E_t[E_{t+1}[x]] = E_t[x] \qquad \text{(L.I.E.)}$$
$$\text{var}\,(x) = E[x^2] - E[x]^2$$
$$\text{var}\,(ax) = a^2 \text{var}\,(x)$$
$$\text{cov}\,(x, y) = E[x\ y] - E[x]E[y]$$
$$\text{cov}\,(ax, y) = a\text{cov}\,(x, y) = \text{cov}\,(x, ay)$$

Expectation of a log-normal

$$E[e^x] = e^{\mu + \frac{1}{2}\sigma^2} \qquad \text{if} \qquad x \sim \mathcal{N}(\mu, \sigma^2)$$

MA(q) processes

If

$$x_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q} \qquad \text{with } \epsilon_t \text{ I.I.D.}$$

Then

$$\text{var}\,(x_t) = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\sigma^2 \qquad \text{(variance)}$$
$$\text{cov}\,(x_t, x_{t-j}) = (\theta_j + \theta_1\theta_{j+1} + \theta_2\theta_{j+2} + \cdots + \theta_{q-j}\theta_q)\sigma^2 \qquad \text{(auto-covariance)}$$
$$\phi_j = \frac{\theta_j + \theta_1\theta_{j+1} + \theta_2\theta_{j+2} + \cdots + \theta_{q-j}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2} \qquad \text{(auto-correlation)}$$

AR(1) process

If

$$x_t = \mu + \rho x_{t-1} + \epsilon_t \qquad \text{with } \epsilon_t \sim \mathcal{N}(0, \sigma),\ \epsilon_t\ I.I.D.$$

Then

$$\text{var}\,(x_t) = \frac{1}{1 - \rho^2}\sigma^2 \qquad \text{(variance)}$$
$$\text{cov}\,(x_t, x_{t-j}) = \frac{\rho^j}{1 - \rho^2}\sigma^2 \qquad \text{(auto-covariance)}$$
$$\phi_j = \rho^j \qquad \text{(auto-correlation)}$$

VR_k statistic

$$VR_k = \frac{1}{k}\frac{\text{var}\,(r_{t,t+k})}{\text{var}\,(r_{t+1})} = 1 + 2\sum_{j=1}^{k-1}\left(1 - \frac{j}{k}\right)\phi_j,$$

where ϕ_j is the jth autocorrelation coefficient of returns.