Solution key: Trial exam 2009

1. (a) See slides.
   (b) $\begin{bmatrix} 5/16 & 5/32 \end{bmatrix}'$
   (c) $E[R^p] = 1.0891, \sigma(R^p) = 0.1398$.
   (d) Expected return would increase (first stock has higher expected return) but also the standard deviation (otherwise the weights could not be optimal in the first place).
   (e) Every investor holds combination of the tangency portfolio and the risk-free asset. Also the tangency portfolio is on the upper part of risky-asset frontier, so it will correspond to the market portfolio. (I wouldn't require the second condition, since it has to be true in equilibrium anyway.)

2. (a) On all nodes we move up $m_{t+1} = 0.99\sqrt{1/1.05}$ on the other nodes, $m_{t+1} = 0.99$.
   (b) $B^{(1)} = E_1[m_2] = 0.9781$. $B^{(2)} = E_1[m_1 m_2] = 0.9566$. (Compute over all possible paths, or, if you spot that consumption growth is IID, just square $B^{(1)}$.
   (c) $0.25 * (0.99)^2 = 0.2450$.
   (d) In the last period, it's values are 10.25, 5, 5 or 0. (Counting from top to bottom), in the second period, its value is 10.25 or 2.41535, in the first period it's value is 6.1471.

3. (a) E.g. noise traders. As we'll see below, here it will be investors overreacting to dividend news.
   (b)

$$\frac{E[r_{t+1}]}{\text{var}\,(r_{t+1})} = \frac{0.05}{(5/4)\sigma_1^2} \qquad \text{and} \qquad \frac{E[r_{t,t+2}]}{\text{var}\,(r_{t,t+2})} = \frac{0.1}{(3/2)\sigma_1^2}$$

   $VR_1 = 1, VR_2 = 3/5$
   (c) On average the second investor should have a higher static allocation to equity, since he faces a better risk/return trade-off. (It is of course better to use a static allocation rule like on your 3rd assignment.)
   (d) $\eta_{r,t+1} = -(1/2)\rho\epsilon_{t+1}$
   (e) $\eta_{d,t+1} = \rho/(1-0.2\rho)\nu_{t+1}$. (See last page.)
   (f)

$$\epsilon_{t+1} = \eta_{d,t+1} - \eta_{r,t+1} = (1-0.2\rho)^{-1}\nu_t + \frac{1}{2}\rho\epsilon_{t+1}$$
$$\Rightarrow \epsilon_{t+1} = \frac{\rho}{(1-0.5\rho)(1-0.2\rho)}\nu_t$$

4. (a) $\min w'\Omega w$, subject to: $w'\iota = 1$ & $w'E[R] = \bar{R}^p$, where $\bar{R}^p$ is the required expected portfolio return.
   (b) $R^f = 1$: wedge with upper part tangent to upper part of parabola. $R^f = 1.1$: wedge with lower part tangent to the lower part of the parabola.
   (c) The two points would have abscissas given by 1 and 1/1.1, respectively. Link: True mean-variance frontier has a slope given by $\sigma(m)/E[m]$, which is also the maximum Sharpe ratio attainable on a portfolio. If we are not sure we have enough data to find the true maximal Sharpe ratio, this turns into the inequality $\sigma(m)/E[m] \geq SR$. For a given R^f, $E[m] = 1/R^f$ and we can solve for $\sigma(m)/E[m]$ through $\sigma(m) = E[m] \cdot SR$
   (d) Left panel: RAF shifts leftwards, which increases the slope of teh MV-frontier. Right panel: HJ-bound shifts upwards.
   (e) If we knew R^f, we would also know $E[m]$, and the HJ-bound in the right panel reduces to a single point.

Appendix: Excessively long solutions for $\eta_{d,t+1}$

As promissed, here is a very detailled solution for the exam question:

For orignal equation $\Delta d_{t+1} = 0.02 + 0.2\Delta d_t + \nu_t$

Summary: The news about dividend growth at time $t+1$ changes expected dividend growth for time $t+1+j$ by $(0.2)^{j-1}\nu_{t+1}$. If you can't see this directly form the equation, the derivation below should help you. Go through it, go back to the equation, and see if you see it now. As soon as you are able to, you can apply your insight directly at the exam without going through all this mess again.

Derivation. You know dividend growth at time $t+1$ already at t, so you will not change expectations about dividend growth at $t+1$. Dividend growth at times $t+2$ and $t+3$ are given by

$$\begin{aligned}\Delta d_{t+1+1} &= 0.02 + 0.2\Delta d_{t+1} + \nu_{t+1} \\ \Delta d_{t+1+2} &= 0.02 + 0.2\Delta d_{t+2} + \nu_{t+2} \\ &= 0.02 + 0.2(0.02) + (0.2)^2\Delta d_{t+1} + 0.2\nu_{t+1} + \nu_{t+2}\end{aligned}$$

By the same pattern

$$\Delta d_{t+1+j} = \underbrace{(1 + 0.2 + 0.2^2 + \cdots 0.2^{j-1})0.02 + (0.2)^j(\Delta d_{t+1})}_{\text{Known at } t} + \underbrace{(0.2)^j\nu_{t+1}}_{\text{Learned at } t+1} + \underbrace{(0.2)^{j-1}\nu_{t+2} + \cdots \nu_{t+j}}_{\text{Learned later}}$$

It should be clear that your change in expectations about dividend growth at time $t+1$ is given by the effect of what is learned at time $t+1$. If not, here is the formal way of showing it. Since $E[\nu_t] = 0$, the expected value of every shock we haven't seen yet is zero and

$$\begin{aligned}E_t[\Delta d_{t+1+j}] &= (1 + 0.2 + 0.2^2 + \cdots 0.2^{j-1})0.02 + (0.2)^j(\Delta d_{t+1}) \\ E_{t+1}[\Delta d_{t+1+j}] &= (1 + 0.2 + 0.2^2 + \cdots 0.2^{j-1})0.02 + (0.2)^j(\Delta d_{t+1}) + (0.2)^j\nu_{t+1}\end{aligned}$$

So,

$$\begin{aligned}E_{t+1}[\Delta d_{t+1}] - E_t[\Delta d_{t+1}] &= 0 \\ E_{t+1}[\Delta d_{t+1+j}] - E_t[\Delta d_{t+1+j}] &= (0.1)^{j-1}\nu_{t+1}\end{aligned}$$

Key insight: There is no change in expectations about anything you know at time t between t and $t+1$. What you knew at time t, you still know at time $t+1$. There is also no change in expectations about future shocks since $E_{t+1}[\nu_{t+1+j}] = E_t[\nu_{t+1+j}] = 0$, for $j \geq 1$. Your update about future dividend growth rates is given by the shock you see at $t+1$ times its influence on dividend growth at time Δd_{t+1+j}, with the AR(1) process, you get the pattern above.

Plugging into the formula for $\eta_{d,t+1}$:

$$\begin{aligned}\eta_{d,t+1} &= (E_{t+1} - E_t)\sum_{j=0}^{\infty}\rho^j\Delta d_{t+1+j} \\ &= \sum_{j=0}^{\infty}\rho^j\left(E_{t+1}[\Delta d_{t+1+j}] - E_t[\Delta d_{t+1+j}]\right) \\ &= 0 + \rho\nu_{t+1} + \rho^2(0.2)\eta_{t+1} + \rho^3(0.2)^2\nu_{t+1} + \cdots \\ &= \rho(1 + (0.2\rho) + (0.2\rho)^2 + (0.2\rho)^3 + \cdots)\nu_{t+1} \\ &= \frac{\rho}{1 - 0.2\rho}\nu_{t+1}\end{aligned}$$

For alternative equation $\Delta d_{t+1} = 0.02 + 0.2\Delta d_t + \nu_{t+1}$

Summary: Same as above, except that news about dividend growth at time $t+1$ changes expected dividend growth for time $t+1+j$ by $(0.2)^j\nu_{t+1}$. If you can't see this directly form the equation, the derivation below should help you. Go through it, go back to the equation, and see if you see it now. As soon as you are able to, you can apply your insight directly at the exam.

Derivation growth at time $t+2$ can be represented as

$$\begin{aligned}\Delta d_{t+1+1} &= 0.02 + 0.2\Delta d_{t+1} + \nu_{t+2}\\ &= 0.02 + 0.2(0.02) + (0.2)^2\Delta d_t + 0.2\nu_{t+1} + \nu_{t+2}\end{aligned}$$

By the same pattern

$$\Delta d_{t+1+j} = \underbrace{(1 + 0.2 + 0.2^2 + \cdots 0.2^j)0.02 + (0.2)^{j+1}(\Delta d_t)}_{\text{Known at } t} + \underbrace{(0.2)^j\nu_{t+1}}_{\text{Learned at } t+1} + \underbrace{(0.2)^{j-1}\nu_{t+2} + \cdots + \nu_{t+j}}_{\text{Learned later}}$$

Again, it should be clear to you that only what you learn at time $t+1$ changes your expectation about future dividend growth at $t+1$. Formally, we see this by first noting that the expected value of every shock we haven't seen yet is zero, so

$$\begin{aligned}E_t[\Delta d_{t+1+j}] &= (1 + 0.2 + 0.2^2 + \cdots 0.2^{j-1})0.02 + (0.2)^{j+1}(\Delta d_t)\\ E_{t+1}[\Delta d_{t+1+j}] &= (1 + 0.2 + 0.2^2 + \cdots 0.2^{j-1})0.02 + (0.2)^{j+1}(\Delta d_t) + (0.2)^j\nu_{t+1}\end{aligned}$$

It follows that

$$\begin{aligned}E_{t+1}[\Delta d_{t+1}] - E_t[\Delta d_{t+1}] &= \nu_{t+1}\\ E_{t+1}[\Delta d_{t+1+j}] - E_t[\Delta d_{t+1+j}] &= (0.1)^j\nu_{t+1}\end{aligned}$$

Plugging into the formula for $\eta_{d,t+1}$:

$$\begin{aligned}\eta_{d,t+1} &= (E_{t+1} - E_t)\sum_{j=0}^{\infty}\rho^j\Delta d_{t+1+j}\\ &= \sum_{j=0}^{\infty}\rho^j\left(E_{t+1}[\Delta d_{t+1+j}] - E_t[\Delta d_{t+1+j}]\right)\\ &= \nu_{t+1} + \rho(0.2\nu_{t+1}) + \rho^2(0.2)^2\eta_{t+1} + \rho^3(0.2)^3\nu_{t+1} + \cdots\\ &= \left(1 + (0.2\rho) + (0.2\rho)^2 + (0.2\rho)^3 + \cdots\right)\nu_{t+1}\\ &= \frac{1}{1-0.2\rho}\nu_{t+1}\end{aligned}$$