Solution key, October 2012

1. (a)

$$\omega = \frac{1}{\lambda}\Omega^{-1}\left[E[R_{t+1}] - R^f\right] = \frac{1}{2}\begin{bmatrix}25 & 0\\ 0 & 16\end{bmatrix}\begin{bmatrix}0.08\\ 0.125\end{bmatrix} = \begin{bmatrix}1\\ 1\end{bmatrix}$$

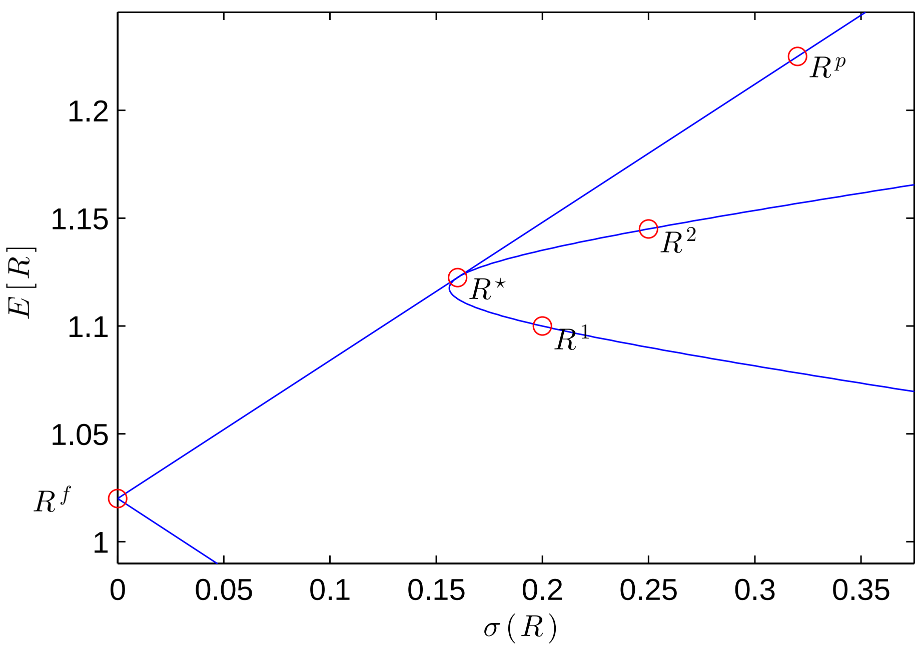
(b)

$$E[R^p] = 1.02 + (1)(0.08) + (1)(0.125) = 1.2250$$
$$\sigma(R^p) = \sqrt{(1)^2(1/5)^2 + (1)^2(1/4)^2} = 0.3202$$

(c) Notice that the investor above allocates 200 % of her wealth to risky assets, so the standard deviation of the return to the portfolio and the expected excess return of her portfolio are both going to be 2 times that of the tangency portfolio, so:

$$E[R^\star] = R^f + 1/2(E[R^p] - R^f) = 1.1225$$
$$\sigma(R^p) = (1/2)\sigma(R^p) = 0.1601$$

(d)



(e)

$$R^p_{t+1} = 1.02 + (1)(0.03) + (1)(0.13) = 1.18$$

2. (a) Both investors put all their wealth into the one risky asset they can invest in, since

$$\omega^1 = \frac{1}{2}(1/25)^{-1}0.08 = 1$$
$$\omega^2 = \frac{1}{2}(1/16)^{-1}0.125 = 1$$

So, investor 1 allocates 100 to stock 1 and investor 2 allocates 100 to stock 2. Since the market capitalization of both stocks is also 100, markets are in equilibrium

(b)

$$\beta^1 = \frac{\text{cov}\left(R^1, R^m\right)}{\text{var}\left(R^m\right)} = \frac{\text{cov}\left(R^1, (0.5)R^1 + (0.5)R^2\right)}{\text{var}\left((0.5)R^1 + (0.5)R^2\right)} = \frac{(1/2)\text{var}\left(R^1\right)}{(0.1601)^2} = 0.7805$$
$$\beta^2 = \frac{\text{cov}\left(R^2, R^m\right)}{\text{var}\left(R^m\right)} = \frac{\text{cov}\left(R^2, (0.5)R^1 + (0.5)R^2\right)}{\text{var}\left((0.5)R^1 + (0.5)R^2\right)} = \frac{(1/2)\text{var}\left(R^2\right)}{(0.1601)^2} = 1.2195$$

So the CAPM predicts that the expected returns to the two stocks should be:

$$E_t[R^1_{t+1} - R^f] = \beta^1 E[R^m_{t+1} - R^f] = 0.7805(1.1225 - 1.02) = 0.08$$
$$E_t[R^2_{t+1} - R^f] = \beta^2 E[R^m_{t+1} - R^f] = 1.2195(1.1225 - 1.02) = 0.125$$

which is just what they are. (No surprise, since the market and tangency portfolios are equal.)

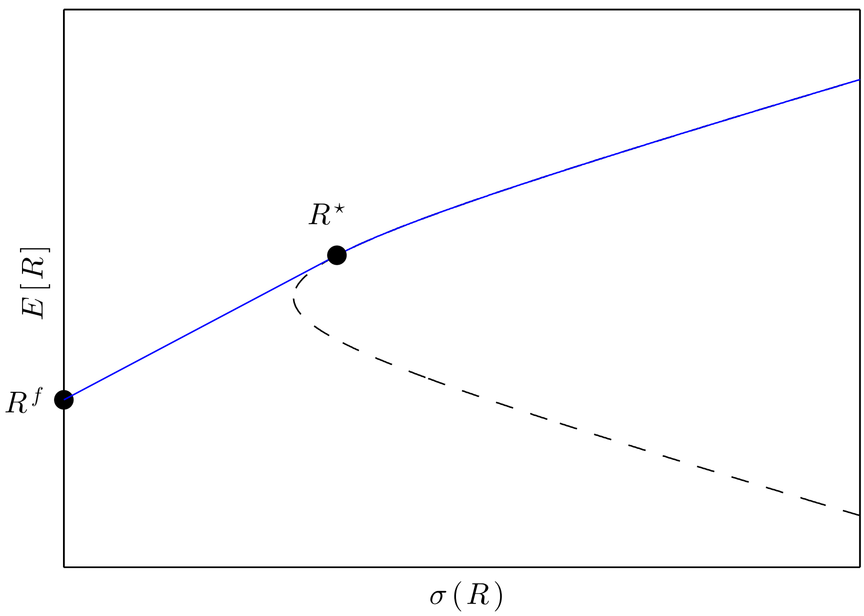
(c) The optimal portfolio weighs of both investors are now:

$$\omega = \frac{1}{\lambda}\Omega^{-1}\left[E[R_{t+1}] - R^f\right] = \frac{1}{2}\begin{bmatrix}25 & 0\\ 0 & 16\end{bmatrix}\begin{bmatrix}0.04\\ 0.0625\end{bmatrix} = \begin{bmatrix}0.5\\ 0.5\end{bmatrix},$$

so both investors would like to allocate 50 to the each Share, which is equal to the supply of each asset.

(d) Higher. The change in expected returns would have to come from somewhere. The natural candidate would be excess demand at old expected returns, investors bid up prices, which reduces expected returns.

3.



From the tangency point onwards, the only way the investor can increase the expected return would be to move along the Risky Asset Frontier. (Which means tilting your portfolio weights towards stocks with higher expected returns.)

4. The Fama-French 3 factor models relates the expected excess return of an asset to exposure to market risk and the SMB and HML factors through

$$E_t[R^i_{t+1} - R^f] = \beta^m_i E_t[R^m_{t+1} - R^f] + \beta^h_i E_t[\text{HML}_{t+1}] + \beta^s_i E_t[\text{SMB}_{t+1}],$$

Assume the expected excess return to the market portfolio is 6 %, the expected return to the HML factor is 4.5 % and the expected return to the SMB factor is 4 %.

(a) First, they sort all traded stocks into quantiles based on their market capitalization. Then they form 2 equal weighted portfolios. One with the stocks from the quantile with smallest market capitalization (Small). The other with the stocks from the quantile with the largest market capitalization (Big). The SMB factor is computed as the excess return to the portfolio of small stocks net of the return to the portfolio of the big stocks (small-minus-big).

(b)

$$E_t[R^i_{t+1} - R^f] = (0.5)(0.06) + (0.5)(0.4) = 0.05$$

(c) This means that the stock has a positive alpha. The way you would exploit that as a mean variance optimizer would be to shift a small part your equity portfolio from the tangency portfolio considering only the factor portfolios into the stock.

5. (a)

$$\widehat{dp} = -1$$

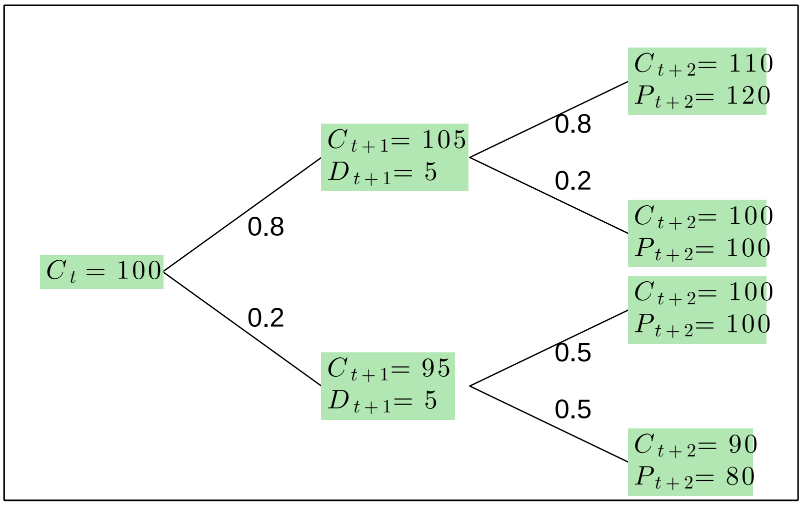
(b)

$$E_t[r_{t+1}] = 0.06 + (0.06)(-1) = 0$$
$$E_t[r_{t+2}] = 0.06 + (0.06)(0.8)(-1) = 0.012$$
$$E_t[r_{t+3}] = 0.06 + (0.06)(0.64)(-1) = 0.0216$$

6. Consider the following simple economy. The probabilities of going up or down depend on the state of the economy. In the upstate at time $t+1$, the probabilities are 0.8 and 0.2, the same as at time t, while in the downstate, both probabilities are 0.5. The representative (typical) investor has a utility function given by

$$U(C) = \frac{1}{2}\sqrt{C},$$

and he has a time discount factor of $\theta = 0.99$.



(a) Taking derivatives of the utility function, we find that:

$$m_{t+1} = \theta \frac{U'(C_{t+1})}{U'(C_t)} = \theta \sqrt{\frac{C_t}{C_{t+1}}}$$

Yielding (from top to bottom)

$$m_{t+1} = \begin{cases} 0.9661, & \text{if up} \\ 1.0157, & \text{if down} \end{cases}$$

$$m_{t+2} = \begin{cases} 0.9672, & \text{if up, up} \\ 1.0144, & \text{if up, down} \\ 0.9649, & \text{if down, up} \\ 1.0171, & \text{if down, down} \end{cases}$$

(b)

$$P_{t+1} = \begin{cases} (0.8)(0.9672)(120) + (0.2)(1.0157)(100) = 113.14, & \text{if up} \\ (0.5)(0.9649)(100) + (0.5)(1.0171)(80) = 88.93 & \text{if down} \end{cases}$$

$$P_t = (0.8)(0.9661)(113.14 + 5) + (0.2)(1.0157)(88.93 + 5) = 110.39$$

(c)

$$E_t[R_{t+1}] = ((0.8)(113.14 + 5) + (0.2)(88.93 + 5))\,/110.39 = 1.0173$$

$$E_{t+1}[R_{t+2}] = \begin{cases} 1.0252 \\ 1.0120 \end{cases}$$

(d)

$$R_t^f = ((0.8)(0.9672) + (0.2)(1.0157))^{-1} = 1.0245$$

$$R_{t+1}^f = \begin{cases} 1.0229 \\ 1.0091 \end{cases}$$

(e) If the the risk-free rate was as high as in the first state, than the representative investor would have wanted to save more (reduce his consumption). This cannot be an equilibrium, since not everyone can save more at the same time. For some investors to be able to save more, some other investors would need to borrow more (increase their consumption), but nobody is going to want to do that in response to a higher interest rate.